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Efficiency Analysis of DMUs Based Separation Hyperplanes in PPS with VRS Technology to Deal with Interval Scale Data

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Abstract

In this paper, in order to evaluate the performance of a DMU in Production Possible Set (PPS) with Variable Return to Scale (VRS) technology, we provide models to obtain non negative weights for inputs for outputs and a nonnegative scalar corresponding to inputs and a nonnegative scalar corresponding to outputs which for the weights and scalars, the number of which DMUs for each one its virtual output plus the scalar corresponding to inputs does not exceed (is less than, if any) its virtual input plus the scalar corresponding to inputs be maximum, provided that for DMU under evaluation, the virtual output plus the scalar corresponding to inputs does not exceed (is less than, if any) the virtual input plus the scalar corresponding to inputs and the virtual input will be positive. We call these weights and scalars the relatively best weight in input-oriented (the relatively strongest weight in input-oriented, if any) for the DMU under evaluation, and if all the weights be positive we call them the best weight in input-oriented (the strongest weight in input-oriented, if any) for the DMU under evaluation. Also, we define input-oriented efficiency and input-oriented strictly efficiency (input-oriented strongly efficiency), respectively, as ratio the number of which DMUs for each one per the relatively best weight in input-oriented and the best weight in input-oriented (the relatively strongest weight in input-oriented), its virtual input plus the scalar related to inputs does not exceed (is less) its virtual input plus the scalar related to outputs, to the total DMUs. Similarly we define the relatively best weight in output-oriented (the relatively strongest weight in input-oriented, if any), the best weight in output-oriented (the strongest weight in output-oriented, if any), output-oriented efficiency and output-oriented strictly efficiency (output-oriented strongly efficiency). The relatively best weight in input-oriented (the relatively strongest weight in input-oriented) indicates normal vector of a superface in the PPS with VRS assumption that the DMU under evaluation is on the superface and the maximum number of which DMUs their performance are no worse than (is better than) the DMU under evaluation separate from the rest of DMUs, with the constraint that the virtual input be positive. Accordingly, we can interpret the rest of the definitions of non-negative weights for inputs and for outputs and nonnegative scalars related to inputs and outputs. In this paper, we present the relationship between these definitions of efficiency with efficiency in the DEA models with VRS assumption.

Keywords: Data envelopment analysis, Efficiency analysis, Separation hyperplanes.

1 | Introduction

The input-oriented BCC Banker et. al [1] multiplier form of Data Envelopment Analysis (DEA), obtains non negative weights for inputs, non-negative weights for outputs and a scalar which is free in sign by maximizing

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virtual output plus a scalar unrestricted in sign, provided that the virtual output plus the scalar does not exceed the virtual input for each DMU and the virtual output be equal to one [1]–[5]. If the scalar be non-negative, it is interpret to subsidization and if the scalar be non-positive, its negative is interpret to setup fee. In this paper, with respect to one's inspiration from the input-oriented. BCC multiplier form, to evaluate the performance of a DMU, we obtain non negative weights for inputs and outputs, a nonnegative scalar corresponding to inputs and a nonnegative scalar corresponding to outputs which for the weights and scalars, the number of which DMUs for each one its virtual output plus the scalar corresponding to inputs does not exceed (is less than, if any) its virtual input plus the scalar corresponding to inputs be maximum, provided that for DMU under evaluation, the virtual output plus the scalar corresponding to inputs does not exceed (is less than, if any) the virtual input plus the scalar corresponding to inputs and the virtual input will be positive [6]–[9].

On the other word, we are going to obtain non negative weights for the inputs and the outputs of DMU under evaluation, a nonnegative scalar related to inputs and a nonnegative scalar related to outputs which for the weights and scalars, the number of which DMUs for each one its income plus subsidization does not exceed (is less than, if any) its cost plus setup fee be maximum, provided that for DMU under evaluation its income plus subsidization will be equal to its cost plus setup fee and cost resulting from the inputs of the DMU will be positive. We call these weights and scalars, the relatively best weight in input-oriented (the relatively strongest weight in input-oriented, if any) for the DMU under evaluation, and if all of these weights be positive, we call them the best weight in input-oriented (the strongest weight in input-oriented, if any) for the DMU under evaluation. Also, we define input-oriented efficiency and input-oriented strictly efficiency (input-oriented strongly efficiency), respectively, as ratio the number of which DMUs for each one per the relatively best weight in input-oriented and the best weight in input-oriented (the relatively strongest weight in input-oriented), its virtual input plus the scalar corresponding to inputs does not exceed (is less) its virtual input plus the scalar corresponding to outputs, to the total DMUs. Similarly, the relatively best weight in output-oriented (the relatively strongest weight in input-oriented, if any), the best weight in output-oriented (the strongest weight in output-oriented, if any), output-oriented efficiency and output-oriented strictly efficiency (output-oriented strongly efficiency) are defined. Also we are going to obtain non negative weights for the inputs and the outputs of DMU under evaluation, a nonnegative scalar corresponding to inputs and a nonnegative scalar related to outputs which for the weights and scalars, the number of which DMUs for each one its income plus subsidization does not exceed (is less than, if any) its cost plus setup fee be maximum, provided that for DMU under evaluation its income plus subsidization will be equal to its cost plus setup fee and both cost resulting from the inputs and income resulting from outputs of the DMU will be positive. We call these weights and scalars the relatively best weight (the relatively strongest weight, if any) for the DMU under evaluation. If all the weights be positive, we call them the best weight (the strongest weight, if any) for the DMU under evaluation. Also, we define efficiency and strictly efficiency (strongly efficiency, if any), respectively, as ratio the number of which DMUs for each one per the relatively best weight in input-oriented and the best weight in input-oriented (the relatively strongest weight, if any), its virtual input plus the scalar related to inputs does not exceed (is less, if any) its virtual input plus the scalar corresponding to outputs, to the total DMUs. The relatively best weight in input-oriented (the relatively strongest weight in input-oriented) indicates normal vector of a superface in the PPS with Variable Return to Scale (VRS) assumption that the DMU under evaluation is on the superface and the maximum number of which DMUs their performance are no worse than (is better than) the DMU under evaluation separate from the rest of DMUs, with the constraint that the virtual input be positive [10], [11]. Accordingly, we can interpret the rest of the definitions of non-negative weights for inputs and for outputs and nonnegative scalars corresponding to inputs and outputs. In this paper, we present the relationship between these definitions of efficiency with efficiency in the DEA models with VRS assumption.

2 | Preliminaries

Suppose we have $n \geq 2$ peer observed DMUs,$\{DMU_j : j = 1,2, \dots, n\}$ which produce multiple outputs $y_{rj}, (r = 1, \dots, s)$, by utilizing multiple inputs $x_{ij}, (i = 1, \dots, m)$. The input and output vectors of DMU_j are denoted by x_j and y_j, respectively, and we assume that $x_j$and y_j are semipositive, i.e. , $x_j \geq 0,\ x_j \neq 0$ and $y_j \geq 0, y_j \neq 0$ for $i = 1, \dots, n$. We use by $(x_j,\ y_j)$ to descript DMU_j, and specially use $(x_o,\ y_o)(o\epsilon\{1,2, \dots, n\})$ as the DMU under evaluation. Throughout this paper, vectors will be denoted by bold letters.

2.1 | The Variable Return to Scale Model

The production set P_v of the BCC model is defined as a set of semi-positive (x, y) as follows [1]:

$$P_v = \left\{(x, y) | \ x \geq \sum_{j=1}^{n} \lambda_j x_j \ \& \ y \leq \sum_{j=1}^{n} \lambda_j y_j \ \& \sum_{j=1}^{n} \lambda_j = 1 \right\},$$

where $(\lambda_1, \dots, \lambda_n)$ is a semi-positive in $\mathbb{R}^n$. The input-oriented BCC model evaluates the efficiency of each $DMU_o$by solving the following linear program:

$$\begin{aligned} \theta^* \quad &= \min \theta, \\ \text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_j \leq \theta x_{o,} \\ & \sum_{j=1}^{n} \lambda_j y_j \geq y_o, \\ & \sum_{j=1}^{n} \lambda_j = 1, \\ & \lambda_j \geq 0, \ \ j = 1, \dots, n, \end{aligned} \qquad (1)$$

where θ is a scaler. Because x_j and y_j are semipositive for $j = 1,2, \dots, n$, $\theta^* > 0$. Also since $(\theta, \lambda = (\lambda_1, \dots, \lambda_n))$ is a feasible solution to *Model (1)*, where $\theta = 1,\ \lambda_j = 0(j \neq o), \boldsymbol{\lambda_o} = 1$, then $\theta^* \leq 1$. Thus $0 < \theta^* \leq 1$. θ^* represents the input-oriented BCC-efficiency value of DMU_o.

Definition 1. (input-oriented BCC-efficient). The performance of DMU_o is the input-oriented BCC-efficient if and only if $\theta^* = 1$.

The dual problem of *Model (1)* is expressed as:

$$\begin{aligned} z^* \quad &= \ \max \ \ u^t y_o + u_o, \\ \text{s.t.} \quad & v^t x_o = 1, \\ & u^t y_j + u_o \leq \ v^t x_j, \qquad j = 1,2, \dots, n, \\ & u \geq o, v \geq o, \end{aligned} \qquad (2)$$

where $v\epsilon\mathbb{R}^m$ and $u\epsilon\mathbb{R}^s$ are row vectors and represent dual variables corresponding to *Eq. (1)* and *Eq. (2)*, respectively. From strong duality theorem $\theta^* = z^*$, thus $0 < z^* \leq 1$.

2.2 | The Two-Phases for Input-Oriented BCC Model

The two-phase process for BCC model evaluates the efficiency of DMU_o by solving the following linear program:

$$
\begin{aligned}
\min \quad & \theta - \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right), \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{io}, i = 1, \dots, m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro}, r = 1, \dots, s, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \geq 0, s_i^- \geq 0, s_r^+ \geq 0, \quad \text{for all } j, \text{for all } i, \text{for all } r,
\end{aligned} \tag{3}
$$

where $\varepsilon > 0$ is the non-Archimedian element.

Definition 2. (BCC-efficient). The performance of DMU_o is BCC-efficient if only if an optimal solution $(\theta^*, \lambda^*, s^{*-}, s^{*+})$ of the two-phase *Model (4)* satisfies $\theta^* = 1, s^{-*} = 0, s^{+*} = 0$.

The dual multiplier form of program *Model (4)* is expressed as:

$$
\begin{aligned}
\max \quad & u^t y_o + u_o,, \\
\text{s.t.} \quad & v^t x_o = 1,, \\
& u^t y_j + u_o \leq v^t x_j, \quad \text{for all } j, \\
& u \geq 1\varepsilon, v \geq 1\varepsilon.
\end{aligned} \tag{4}
$$

By *Definition 2* and by strong duality theorem, the performance of DMU_o is BCC-efficient if only if an optimal solution (u^*, v^*) of *Model (4)* satisfies $u^{*t} y_o + u_o = 1$.

Definition 3. (Reference set) reference set of DMU_o denoted by E_o is defined as:

$E_o = \{DMU_j | j \epsilon \{1, \dots, n\} \& \lambda_j^* > 0 \text{ in some optimal solution } (\theta^*, \lambda^*, s^{-*}, s^{+*}) \text{ of } Model\ (3)\}$.

Theorem 1. The DMUs in E_o are BCC-efficient.

Proof: see [4].

Definition 4. (extreme BCC-efficient) DMU_o is extreme BCC-efficient if only if $E_o = \{DMU_o\}$.

Theorem 2. If DMU_o be extreme BCC-efficient, then DMU_o is BCC-efficient

Proof: see [4].

Theorem 3. DMU_o is extreme BCC-efficient iff has an optimal objective function valve of one.

$$
\begin{aligned}
\min \quad & \theta - \varepsilon \sum_{j \neq o} \lambda_j, \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{io}, i = 1, \dots, m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro}, r = 1, \dots, s, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \geq 0, s_i^- \geq 0, s_r^+ \geq 0 \quad \text{for all } j, \text{for all } i, \text{for all } r,
\end{aligned} \tag{5}
$$

Proof: let $\mathrm{DMU_o}$ not be extreme BCC-efficient. Then, there exists an optimal solution $(\theta^*, \lambda^*, s^{*-}, s^{*+})$ of *Model (2)* such that a $\lambda_j^* > 0 (j \neq o)$. Also, since $(\theta, \lambda = (\lambda_1, \dots, \lambda_n))$ is a feasible solution to *Model (4)*, where $\theta = 1$, $\lambda_j = 0 (j \neq o), \lambda_o = 1$, thus $\theta^* \leq 1$. Therefore $\theta^* - \varepsilon \sum_{j \neq o} \lambda^*{}_j < 1$. Let the solution objective function valve of *Model (2)* and *(4)* is less one, and let $(\tilde{\theta}, \tilde{\lambda})$ is an optimal solution of the model, then either $\tilde{\theta} < 1$ or ($\tilde{\theta} = 1$ and $\sum_{j \neq o} \tilde{\lambda_j} > 0$). If $\tilde{\theta} < 1$, $\mathrm{DMU_o}$ isn't extreme BCC-efficient. If $\tilde{\theta} = 1$ and $\sum_{j \neq o} \tilde{\lambda_j} > 0$, then either $(\tilde{s}^-, \tilde{s}^+) \neq (0,0)$ or $(\tilde{s}^-, \tilde{s}^+) = (0,0)$, where $\tilde{s}^- = \tilde{\theta}, x_o - \sum_{j=1}^{n} \tilde{\lambda}_j x_j$, and $\tilde{s}^+ = \sum_{j=1}^{n} \tilde{\lambda}_j y_j - y_o$. If $(\tilde{s}^-, \tilde{s}^+) \neq (0,0)$, since$\left(\tilde{\theta}, \tilde{\lambda}, \tilde{s}^-, \tilde{s}^+\right)$ is a feasible solution of *Model (2)*, thus $\mathrm{DMU_o}$ isn't BCC-efficient, therefore $\mathrm{DMU_o}$ isn't extreme BCC-efficient. If $(\tilde{s}^-, \tilde{s}^+) = (0,0)$, then either $\left(\tilde{\theta}, \tilde{\lambda}, \tilde{s}^-, \tilde{s}^+\right)$ is an optimal solution of *Model (2)* or isn't. If $\left(\tilde{\theta}, \tilde{\lambda}, \tilde{s}^-, \tilde{s}^+\right)$ be an optimal solution of *Model (2)*, since $\sum_{j=1}^{n} \tilde{\lambda}_j > 0$, thus $\mathrm{DMU_o}$ isn't extreme BCC-efficient. If $\left(\tilde{\theta}, \tilde{\lambda}, \tilde{s}^-, \tilde{s}^+\right)$ not be an optimal solution of *Model (2)*, then there exists an optimal solution$(\theta^*, \lambda^*, s^{*-}, s^{*+})$ of *Model (2)* such that $\theta^* = 1$ and $(s^{*-}, s^{*+}) \neq (0,0)$, thus $\mathrm{DMU_o}$ isn't extreme BCC-efficient.

3 | Efficiency Analysis of DMUs based Separation Hyperplanes in PPS with Variable Return to Scale Technology

Definition 5. Let $\Lambda_v \subset \mathbb{R}^{m+s+1}$ be

$$\Lambda_v = \{(u, v, u_o, v_o) | u \epsilon \mathbb{R}^s \;\&\; \mathbf{v} \epsilon \mathbb{R}^m \;\&\; u_o \epsilon \mathbb{R}^{\geq 0} \& \; v_o \epsilon \mathbb{R}^{\geq 0} \& \; (u, v) \geq (0,0)\}. \tag{6}$$

We define a map $f_v : \Lambda_v \to \mathbb{N} \cup \{0\}$

by

$$f_v(u, v, u_o, v_o) = \left|\left\{\mathrm{DMU_j} \middle| j \epsilon \{1,2, \dots, n\} \& v^t x_j + v_o \geq u^t y_j + u_o\right\}\right|, \tag{7}$$

where Λ_v defined by *Eq. (6)*.

Definition 6. Let $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o) \epsilon \Lambda_v$. We say $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ is relatively best weight in input-oriented (in output-oriented) in Λ_v for $\mathrm{DMU_o}$ if

$$\bar{v}^t x_o + v_o = \bar{u}^t y_o + u_o \& \; \bar{v}^t x_o > 0 \; (\bar{u}^t y_o > 0),$$

and

$$\text{for all } (u, v, u_o, v_o) \Big((u, v, u_o, v_o) \epsilon \Lambda_c \;\&\; v^t x_o > 0 \; (u^t y_o > 0) \;\&\; v^t x_o + v_o = u^t y_o + u_{o,} \implies f_v(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o) \;\; \geq f_v(u, v, u_o, v_o)\Big).$$

Definition 7. Let $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o) \epsilon \Lambda_v$. We say $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ is relatively best weight in Λ_v for $\mathrm{DMU_o}$ if

$$\bar{v}^t x_o + v_o = \bar{u}^t y_o + u_o \;\&\; \bar{v}^t x_o > 0 \;\&\; \bar{u}^t y_o > 0.$$

And

$$\text{for all } (u, v, u_o, v_o)\big((u, v, u_o, v_o) \epsilon \Lambda_c \;\&\; v^t x_o + v_o = u^t y_o + u_o \& \; \bar{v}^t x_o > 0 \;\&\; \bar{u}^t y_o > 0 \implies f_v(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o) \geq f_v(u, v, u_o, v_o)\big).$$

Definition 8. Let $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o) \epsilon \Lambda_v$. We say $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ is best weight in Λ_v *for* $\mathrm{DMU_o}$ if

$$\bar{v}^t x_o + v_o = \bar{u}^t y_o + u_o \& \; (\bar{u}, \bar{v}) > (0,0).$$

Definition 9. Let $(\bar{u},\bar{v},\bar{u}_o,\bar{v}_o)\epsilon\Lambda_v$. We say (u,v,u_o,v_o) is relatively strongest weight in input-oriented (in output-oriented) in Λ_v for DMU_o if

$$\bar{v}^t x_o + v_o = \bar{u}^t y_o + u_o \& \ \bar{v}^t x_o > 0 \ (\bar{u}^t y_o > 0) \ \& \ g_v(\bar{u},\bar{v},\bar{u}_o,\bar{v}_o) \geq 1,$$

and

$$\text{for all } (u,v,u_o,v_o)\big((u,v,u_o,v_o)\epsilon\Lambda_v \ \& \ v^t x_o > 0 \ (u^t y_o > 0) \ \& \ v^t x_o + v_o = u^t y_o + u_o \Longrightarrow g_v(\bar{u},\bar{v},\bar{u}_o,\bar{v}_o) \geq g_v(u,v,u_o,v_o)\big).$$

Definition 10. Let $(\bar{u},\bar{v},\bar{u}_o,\bar{v}_o)\epsilon\Lambda_c$. We say $(\bar{u},\bar{v},\bar{u}_o,\bar{v}_o)$ is relatively strongest weight in Λ_v for DMU_o if

$$\bar{v}^t x_o + v_o = \bar{u}^t y_o + u_o \& \ \bar{v}^t x_o > 0 \ \& \ \bar{u}^t y_o > 0 \ \& \ g_v(\bar{u},\bar{v},\bar{u}_o,\bar{v}_o) \geq 1,$$

and

$$\text{for all } (u,v,u_o,v_o)\big((u,v,u_o,v_o)\epsilon\Lambda_v \ \& \ v^t x_o > 0 \ u^t y_o > 0 \ \& \ v^t x_o + v_o = u^t y_o + u_o \Longrightarrow g_v(\bar{u},\bar{v},\bar{u}_o,\bar{v}_o) \geq g_v(u,v,u_o,v_o)\big).$$

Remark 1. Since x_j and y_j are semipositive, it follows $\sum_{r=1}^{s} y_{ro} > 0, \sum_{i=1}^{m} x_{io} > 0$. Now if $\sum_{r=1}^{s} y_{ro} = \sum_{i=1}^{m} x_{io}$, by taking $v^t = (1,\dots,1)\epsilon\mathbb{R}^m$, $u^t = (1,\dots,1)\epsilon\mathbb{R}^s$, $u_o = 0$ and $v_o = 0$, if $\sum_{r=1}^{s} y_{ro} > \sum_{i=1}^{m} x_{io}$, by taking $\alpha = \left(\sum_{r=1}^{s} y_{ro} \middle/ \sum_{i=1}^{m} x_{io}\right)$, $u^t = \alpha(1,\dots,1)\epsilon\mathbb{R}^s$, $v^t = \alpha(1,\dots,1)\epsilon\mathbb{R}^m$, $u_o = 0$ and $v_o = 0$, and finally if $\sum_{r=1}^{s} y_{ro} < \sum_{i=1}^{m} x_{io}$, by taking $\beta = \left(\sum_{i=1}^{m} x_{io} \middle/ \sum_{r=1}^{s} y_{ro}\right)$, $u^t = \beta(1,\dots,1)\epsilon\mathbb{R}^s$, $v^t = \beta(1,\dots,1)\epsilon\mathbb{R}^m$ $u_o = 0$ and $v_o = 0$, we have $v^t x_o + v_o = u^t y_o + u_o$, $v^t x_o > 0 \ u^t y_o > 0$, $u \geq 1\varepsilon$, $v \geq 1\varepsilon$.

This shows that there is not any relatively strongest weight in input-oriented (in output-oriented) in Λ_v for DMU_o if

$$\text{for all } (u,v,u_o,v_o)\big((u,v,u_o,v_o)\epsilon\Lambda_v \ \& \ v^t x_o > 0 \ (u^t y_o > 0) \ \& \ v^t x_o + v_o = u^t y_o + u_o \Longrightarrow g_v(u,v,u_o,v_o) = 0\big).$$

And there is not any relatively strongest weight in Λ_c for DMU_o if

$$\text{for all } (u,v,u_o,v_o)\big((u,v,u_o,v_o)\epsilon\Lambda_v \ \& \ v^t x_o > 0 \ u^t y_o > 0 \ \& \ v^t x_o + v_o = u^t y_o + u_o \Longrightarrow g_v(u,v,u_o,v_o) = 0\big).$$

Also there is not any strongest weight in Λ_v for DMU_o if

$$\text{for all } (u,v,u_o,v_o)\big((u,v,u_o,v_o)\epsilon\Lambda_c \ \& \ (u,v) > (0,0) \ \& \ v^t x_o + v_o = u^t y_o + u_o \Longrightarrow g_v(u,v,u_o,v_o) = 0\big).$$

Definition 10 (input-oriented (output-oriented) Λ_c-efficiency). If $(\bar{u},\bar{v},\bar{u}_o,\bar{v}_o)$ be relatively best weight in input-oriented (output-oriented) in Λ_c for DMU_o , then

input-oriented (output-oriented) Λ_v-efficiency of

$$DMU_o = \frac{f_v(\bar{u},\bar{v},\bar{u}_o,\bar{v}_o)}{n}$$

Definition 11 (input-oriented (output-oriented) Λ_c-efficient). DMU_o is said to be in input-oriented (in output-oriented) Λ_v-efficient if input-oriented (output-oriented) Λ_c-efficiency of DMU_o = 1.

Definition 12 (Λ_c-efficiency). If $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ be relatively best weight in Λ_c for DMU_o, then efficiency

$$\Lambda_v = \frac{f_v(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)}{n}.$$

Definition 13 (Λ_v-efficient). DMU_o is said to be Λ_v-efficient if Λ_c-efficiency = 1.

Definition 14 (strictly Λ_v-efficiency). If $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ be best weight in Λ_v, for DMU_o then strictly Λ_v-efficiency = $\frac{f_v(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)}{n}$.

Definition 15 (strictly Λ_v-efficient). DMU_o is said to be strictly Λ_v-efficient if strictly Λ_v-efficiency of DMU_o = 1.

Definition 16 (input-oriented (output-oriented) strongly Λ_c-efficiency). If there is a $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ relatively strongest weight in input-oriented (output-oriented) in Λ_v for DMU_o, then input-oriented (output-oriented) strongly Λ_v-efficiency = $\frac{g_v(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)}{n-1}$.

Definition 17. If there is not any relatively strongest weight in input-oriented (output-oriented) in Λ_v for DMU_o, then input-oriented (output-oriented) strongly Λ_c-efficiency = 0.

Definition 18 (input-oriented (output-oriented) strongly Λ_v-efficient). DMU_o is said to be input-oriented (output-oriented) strongly Λ_v-efficient if input-oriented (output-oriented)strongly Λ_v-efficiency of DMU_o = 1.

Definition 19 (strongly Λ_c-efficiency). If there is a $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ relatively strongest weight in Λ_v for DMU_o Then strongly Λ_v-efficiency = $\frac{g_v(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)}{n-1}$.

Definition 20. If there is not any relatively strongest weight in Λ_v for DMU_o, then strongly Λ_v-efficiency = 0.

Definition 21 (strongly Λ_v-efficient). DMU_o is said to be strongly Λ_v-efficient if strongly Λ_v-efficiency of DMU_o=1.

Proposition 1. Let $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ be relatively best weight in input-oriented (output-oriented) in Λ_v for DMU_o, and let $(\bar{u}, \bar{v}) > (0,0)$. Then $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ is best weight in Λ_v for DMU_o.

Proof: if $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ not be best weight in Λ_v for DMU_o, then by *Definition 7*, there is some $(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o) \epsilon \Lambda_v$ such that $(\tilde{u}, \tilde{v}) > (0,0)$, $\tilde{v}^t x_o + \tilde{v}_o = \tilde{u}^t y_o + \tilde{u}_o$ and $f_v(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o) < f_v(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o)$. On the other hand, since x_j and y_j are semipositive, $\tilde{v}^t x_o + \tilde{v}_o > 0$ $(\tilde{u}^t y_o + \tilde{u}_o > 0)$. Therefore, noting that $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ is relatively best weight in input-oriented (output-oriented) in Λ_c for DMU_o,.$f_v(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o) \geq f_v(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o)$ which is contradiction with this fact that $f_v(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o) < f_v(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o)$. Thus $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ is best weight in Λ_v for DMU_o.

Proposition 2. If $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ be relatively best weight in Λ_v for DMU_o, and if $(\bar{u}, \bar{v}) > (0,0)$. Then $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ is best weight in Λ_v for DMU_o.

Proof: similar to the proof of *Proposition 1*.

Proposition 3. If there is a $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ relatively strongest weight in input-oriented (output-oriented) in Λ_v for DMU_o, and if $\bar{u}^t y_o > 0$ $(\bar{v}^t x_o > 0)$. Then $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ is relatively best weight in Λ_c for DMU_o.

Proof: if $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ be relatively strongest weight in input-oriented (output-oriented) in Λ_c for DMU_o, and if $\bar{u}^t y_o > 0$ $(\bar{v}^t x_o > 0)$, then

$$\text{for all } (u, v, u_o, v_o)\big((u, v, u_o, v_o) \epsilon \Lambda_v \;\&\; v^t x_o > 0\ (u^t y_o > 0) \;\&\; v^t x_o + v_o = u^t y_o + u_o \Longrightarrow g_v(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o) \geq g_v(u, v, u_o, v_o)\big). \tag{8}$$

Now we assume $(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o) \epsilon \Lambda_v$ be relatively best weight in Λ_v for $\mathrm{DMU_o}$, then

$$\tilde{v}^t x_o > 0, \;\; \tilde{u}^t y_o > 0, \;\; \tilde{v}^t x_o + \tilde{v}_o = \tilde{u}^t y_o + \tilde{u}_o,$$

and

$$f_v(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o) \le f_v(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o). \tag{9}$$

Also, by *Eq. (8)*, we have

$$g_v(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o) \ge g_v(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o). \tag{10}$$

On the other hand, by *Definition 1* and *Definition 2*, we have

$$f_v(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o) \ge g_v(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o),$$

and

$$f_v(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o) \ge g_v(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o). \tag{11}$$

By *Eqs. (1)-(4)*, we have

$$f_v(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o) = f_v(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o) = g_v(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o) = g_v(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o).$$

Thus $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ is relatively best weight in Λ_c for $\mathrm{DMU_o}$.

Proposition 4. If there is a $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ relatively strongest weight in input-oriented (output-oriented) in Λ_v for $\mathrm{DMU_o}$, and if $\bar{u}^t y_o > 0$ ($\bar{v}^t x_o > 0$). Then $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ is relatively strongest weight in Λ_v for $\mathrm{DMU_o}$.

Proof: similar to the proof of *Proposition 3*.

Proposition 5. If there is a $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ relatively strongest weight in input-oriented (output-oriented) in Λ_v for $\mathrm{DMU_o}$, and if $(\bar{u}, \bar{v}) > (0,0)$, Then $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ is strongest weight in Λ_v for $\mathrm{DMU_o}$.

Proof: Similar to the proof of *Proposition 3*.

Proposition 6. If there is a $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ relatively strongest weight in Λ_c for $\mathrm{DMU_o}$, and if $(\bar{u}, \bar{v}) > (0,0)$, Then $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ is strongest weight in Λ_v for $\mathrm{DMU_o}$.

Proof: Similar to the proof of *Proposition 3*.

Theorem 1. Let $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ be relatively best weight in input-oriented (output-oriented) in Λ_c for $\mathrm{DMU_o}$, let $p = f_v(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ let

$$\left\{\mathrm{DMU}_{j_1}, \dots, \mathrm{DMU}_{j_p}\right\} = \left\{\mathrm{DMU}_j \middle| j \epsilon \{1, \dots, n\} \& \bar{v}^t x_j + \bar{v}_o \ge \bar{u}^t y_j + \bar{u}_o\right\}, \tag{12}$$

and let $\bar{t} = (\bar{t}_1, \dots, \bar{t}_n)$ with

$$\begin{cases} 0, & j \epsilon \{j_1, \dots, j_p\}, \\ 1, & j \epsilon \{1, \dots, n\} - \{j_1, \dots, j_p\}. \end{cases} \tag{13}$$

Then $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o, \bar{t})$ is an optimal solution of the following model:

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{n} t_j, \\
\text{s.t.} \quad & v^t x_o + v_o - (u^t y_o + u_o) = 0, \;\; v^t x_o \geq \varepsilon \; (u^t y_o \geq \varepsilon), \\
& v^t x_j + v_o - (u^t y_j + u_o) + M t_j \geq 0, \;\; \text{for all } j \;\; (M \gg 0), \\
& u \geq 0, \;\; v \geq 0, \; u_o \geq 0, \; v_o \geq 0, \;\; t_j \in \{0,1\}.
\end{aligned}
$$

Conversely, if $(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o, \tilde{t})$ be an optimal solution of *Model (4)*, then $(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o)$ is a relatively best weight in input-oriented (output-oriented) in Λ_c for $\mathrm{DMU_o}$.

Proof: since $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ is a relatively best weight in input-oriented (output-oriented) in Λ_c for $\mathrm{DMU_o}$, then, by *Eqs. (5)* and *(6)*, $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o, \bar{t})$ is a feasible solution of *Model (4)*. On the other hand, since $(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o, \tilde{t})$ is an optimal solution for *Model (4)*, we have $n - \sum_{j=1}^{n} \tilde{t}_j \geq n - \sum_{j=1}^{n} \bar{t}_j$, therefore $f_v(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o) \geq f_v(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$. Also $f_c(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o) \leq f_c(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$, since $(\bar{u}, \bar{v})$ is relatively best weight in input-oriented (output-oriented) in Λ_c for $\mathrm{DMU_o}$.

Hence,

$$f_v(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o) = f_v(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o) = n - \textstyle\sum_{j=1}^{n} \bar{t}_j = n - \sum_{j=1}^{n} \tilde{t}_j.$$

Thus $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o, \bar{t})$ is an optimal solution to *Model (4)*, and $(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o)$ is relatively best weight in input-oriented (output-oriented) in Λ_c for $\mathrm{DMU_o}$.

Theorem 2. Let $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o, \bar{t})$ be an optimal solution of the following model:

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{n} t_j, \\
\text{s.t.} \quad & v^t x_o + v_o - (u^t y_o + u_o) = 0, \;\; v^t x_o = 1 \; (u^t y_o = 1), \\
& v^t x_j + v_o - (u^t y_j + u_o) + M t_j \geq 0, \;\; \text{for all } j \;\; (M \gg 0), \\
& u \geq 0, \;\; v \geq 0, \; u_o \geq 0, \; v_o \geq 0, \;\; t_j \in \{0,1\}.
\end{aligned}
\tag{14}
$$

Then $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ is relatively best weight in input-oriented (output-oriented) in Λ_c for $\mathrm{DMU_o}$.

Proof: let $(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o)$ be relatively best weight in input-oriented (output-oriented) in Λ_c for $\mathrm{DMU_o}$, let $p = f_c(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o)$, let

$$\left\{\mathrm{DMU}_{j_1}, \dots, \mathrm{DMU}_{j_p}\right\} = \left\{\mathrm{DMU}_j \middle| j \in \{1, \dots, n\} \,\&\, \tilde{v}^t x_j + \tilde{v}_o \geq \tilde{u}^t y_j + \tilde{u}_o\right\},$$

Then

$$\tilde{v}^t x_o + \tilde{v}_o = \tilde{u}^t y_o + \tilde{u}_o, \tilde{v}^t x_o > 0 \; (\tilde{u}^t y_o > 0).$$

So that by taking $k = \tilde{v}^t x_o (k = \tilde{u}^t y_o)$, $\hat{u} = {}^{\tilde{u}}/_{k}$, $\hat{v} = {}^{\tilde{v}}/_{k}$, $\hat{u}_o = {}^{\tilde{u}_o}/_{k}$, and $\hat{v}_o = {}^{\tilde{v}_o}/_{k}$, we have

$$
\begin{aligned}
& \tilde{v}^t x_o > 0 \; (\tilde{u}^t y_o > 0), \;\; (\hat{u}, \hat{v}) \geq (0,0), \;\; \hat{v}^t x_o + \hat{v}_o = (\hat{u}^t y_o + \hat{u}_o), \;\; \hat{v}^t x_{j_i} + \hat{v}_o - \\
& \left(\hat{u}^t y_{j_i} + \hat{u}_o\right) \geq 0, i = 1, \dots, p.
\end{aligned}
\tag{15}
$$

Thus $(\hat{u}, \hat{v}, \hat{u}_o, \hat{v}_o, \hat{t})$ is a feasible solution for *Model (15)*, where $\hat{t} = (\hat{t}_1, \dots, \hat{t}_n)$ with

$$\hat{t}_j = \begin{cases} 0 & j \in \{j_1, \dots, j_p\}. \\ 1 & j \in \{1, \dots, n\} - \{j_1, \dots, j_p\}' \end{cases}$$

Therefore, sine $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o, \bar{t})$ is an optimal solution for *Model (5)*, $n - \sum_{j=1}^{n} \hat{t}_j \leq n - \sum_{j=1}^{n} \bar{t}_j$. Hence

$$f_c(\hat{u},\hat{v},\hat{u}_o,\hat{v}_o) \le f_c(\bar{u},\bar{v},\bar{u}_o,\bar{v}_o). \tag{16}$$

Also by *Model (15)*

$$f_c(\tilde{u},\tilde{v},\tilde{u}_o,\tilde{v}_o) \le f_v(\hat{u},\hat{v},\hat{u}_o,\hat{v}_o). \tag{17}$$

On the other hand, since $(\tilde{u},\tilde{v},\tilde{u}_o,\tilde{v}_o)$ is relatively best weight in input-oriented (output-oriented) in Λ_c for DMU_o and

$$\tilde{v}^t x_o + \tilde{v}_o = \tilde{u}^t y_o + \tilde{u}_o, \tilde{v}^t x_o > 0 \ (\tilde{u}^t y_o > 0).$$

We have

$$f_c(\tilde{u},\tilde{v},\tilde{u}_o,\tilde{v}_o) \ge f_c(\bar{u},\bar{v},\bar{u}_o,\bar{v}_o). \tag{18}$$

Thus, by *Models (16)-(18)*,

$$f_c(\tilde{u},\tilde{v},\tilde{u}_o,\tilde{v}_o) = f_c(\bar{u},\bar{v},\bar{u}_o,\bar{v}_o),$$

Therefore $(\bar{u},\bar{v},\bar{u}_o,\bar{v}_o)$ is relatively best weight in input-oriented (output-oriented) in Λ_c for DMU_o.

Corollary 1. Let $(\bar{u},\bar{v},\bar{u}_o,\bar{v}_o)$ be an optimal solution of *Model (5)*, then

input-oriented (output-oriented) Λ_v-efficiency of

$$DMU_o = \frac{f_c(\bar{u},\bar{v},\bar{u}_o,\bar{v}_o)}{n} = \frac{n-\sum_{j=1}^{n}\bar{t}_j}{n}.$$

Proof: *Theorem 2*.

Corollary 2. DMU_o is input-oriented (output-oriented) Λ_v-efficient if only if the optimal objective function value of *Model (5)* is zero.

Proof: *Theorem 2*.

Corollary 3. If there is some optimal solution$(\bar{u},\bar{v},\bar{u}_o,\bar{v}_o)$ for *Model (5)* such that $\bar{v}^t x_o > 0$ $(\bar{u}^t y_o > 0)$, then $(\bar{u},\bar{v},\bar{u}_o,\bar{v}_o)$ is relatively best weight in Λ_c for DMU_o.

Proof: *Proposition 1* and *Theorem 2*.

Corollary 4. If there is some optimal solution$(\bar{u},\bar{v},\bar{u}_o,\bar{v}_o)$ for *Model (5)* such that $(\bar{u},\bar{v}) > (0,0)$, then $(\bar{u},\bar{v},\bar{u}_o,\bar{v}_o)$ is best weight in Λ_c for DMU_o.

Proof: *Proposition 2* and *Theorem 2*.

Corollary 5. If there is some optimal solution$(\bar{u},\bar{v},\bar{t})$ for *Model (2)* such that $(\bar{u},\bar{v}) > (0,0)$, and $\sum_{j=1}^{n}\bar{t}_j = 0$, then DMU_o is strictly Λ_c-efficient.

Proof: *Proposition 1* and *Theorem 2*.

Theorem 3. Let $(\tilde{u},\tilde{v},\tilde{u}_o,\tilde{v}_o,\tilde{t})$ be an optimal solution of the following model:

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} t_j, \\ \text{s.t.} \quad & v^t x_o + v_o - (u^t y_o + u_o) = 0, \ \ v^t x_o \ge \varepsilon, \ \ u^t y_o \ge \varepsilon, \\ & v^t x_j + v_o - (u^t y_j + u_o) + M t_j \ge 0, \ \ \text{for all } j \ \ (M \gg 0), \\ & u \ge 0, \ \ v \ge 0, \ u_o \ge 0, \ v_o \ge 0, \ \ t_j \in \{0,1\}. \end{aligned}$$

Then $(\tilde{u},\tilde{v},\tilde{u}_o,\tilde{v}_o)$ is a relatively best weight in Λ_c for DMU_o

Conversely, let $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ be relatively best weight in Λ_c for DMU_o, Then $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o, \bar{t})$, where $\bar{t}$ is defined by *Models (5)* and *(6)* is an optimal solution of *Model (3)*.

Proof: similar to the proof of *Theorem 1*.

Theorem 4. Let $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o, \bar{t})$ be an optimal solution of the following model:

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} t_j, \\ \text{s.t.} \quad & v^t x_o + v_o - (u^t y_o + u_o) = 0, \;\; v^t x_o \geq 1, \;\; u^t y_o \geq 1, \\ & v^t x_j + v_o - (u^t y_j + u_o) + M t_j \geq 0\,, \;\; \text{for all } j \;\; (M \gg 0), \\ & u \geq 0, \;\; v \geq 0, \; u_o \geq 0, \; v_o \geq 0, \;\; t_j \epsilon \{0,1\}. \end{aligned}$$

Then $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ is relatively best weight in input-oriented (output-oriented) in Λ_c for DMU_o.

Proof: similar to the proof of *Theorem 2*.

Corollary 6. Let $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ be an optimal solution of *Model (4)*, then Λ_v-efficiency of

$$DMU_o = \frac{f_c(\bar{\mathbf{u}}, \bar{\mathbf{v}}, \bar{u}_o, \bar{v}_o)}{n} = \frac{n - \sum_{j=1}^{n} \bar{t}_j}{n}.$$

Proof: *Theorem 4*.

Corollary 7. DMU_o is Λ_v-efficient if only if the optimal objective function value of *Model (4)* is zero.

Proof: *Theorem 4*.

Corollary 8. If there is some optimal solution$(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ for *Model (4)* such that $(\bar{u}, \bar{v}) > (0,0)$, then $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ is best weight in Λ_c for DMU_o.

Proof: *Proposition 2* and *Theorem 4*.

Corollary 9. If there is some optimal solution$(\bar{u}, \bar{v}, \bar{t})$ for *Model (5)* such that $(\bar{u}, \bar{v}) > (0,0)$, and $\sum_{j=1}^{n} \bar{t}_j = 0$, then DMU_o is strictly Λ_c-efficient.

Proof: *Proposition 1* and *Theorem 4*.

Theorem 5. Let $(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o, \tilde{t})$ be an optimal solution of the following model:

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} t_j, \\ \text{s.t.} \quad & v^t x_o + v_o - (u^t y_o + u_o) = 0, \\ & v^t x_j + v_o - (u^t y_j + u_o) + M t_j \geq 0\,, \;\; \text{for all } j \;\; (M \gg 0), \\ & u \geq 1\varepsilon, \;\; v \geq 1\varepsilon, \; u_o \geq 0, \; v_o \geq 0, \;\; t_j \epsilon \{0,1\}. \end{aligned}$$

Then $(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o)$ is a relatively best weight in Λ_v for DMU_o.

Conversely let $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ be best weight in Λ_c for DMU_o. Then $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o, \bar{t})$, where $\bar{t}$ is defined by *Models (5)* and *(6)*, is an optimal solution of *Model (5)*.

Proof: by *Remark 1* there is a $(u, v, u_o, v_o) \epsilon \Lambda_v$ such that

$$v^t x_o + v_o = u^t y_o + u_o, \;\; u \geq 1\varepsilon, \;\; v \geq 1\varepsilon.$$

Thus (u, v, u_o, v_o, t), where t is defined by *Models (5)* and *(6)*, is a solution feasible for *Model (6)*.

Let $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ be best weight in Λ_v for DMU_o, let $p = f_c(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$, and let

$$\left\{DMU_{j_1}, \dots, DMU_{j_p}\right\} = \{DMU_j | j\epsilon\{1, \dots, n\} \& \bar{v}^t x_o \geq \bar{u}^t y_o\},$$

Then

$$(\bar{u}, \bar{v}) > (0,0), \ \ \bar{v}^t x_o + \bar{v}_o = \bar{u}^t y_o + \bar{u}_o, \ \bar{v}^t x_{j_i} + \bar{v}_o - (\bar{u}^t y_{j_i} + \bar{u}_o) \geq 0, \ \ i = 1, \dots, p.$$

Thus by taking $k = \min\left\{\min_r\{\bar{u}_r\}, \min_i\{\bar{v}_i\}\right\}$ we have

$$(\bar{u}, \bar{v}) \geq k(1,1), \ \ \bar{v}^t x_o + \bar{v}_o = \bar{u}^t y_o + \bar{u}_o, \ \bar{v}^t x_{j_i} + \bar{v}_o - \left(\bar{u}^t y_{j_i} + \bar{u}_o\right) \geq 0, i = 1, \dots, p.$$

Thus $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o, \bar{t})$, where $\bar{t}$ is defined by *Model (6)* is a feasible solution for *Model (5)*, and since $(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o, \tilde{t})$ is an optimal solution of the model, therefore $n - \sum_{j=1}^{n} \tilde{t}_j \geq n - \sum_{j=1}^{n} \bar{t}_j$. Hence

$$f_c(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o, \tilde{t}) \geq f_c(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o),$$

Also

$$f_c(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o, \tilde{t}) \leq f_c(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o),$$

Since $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ is a best weight in Λ_c for DMU_o. Thus

$$f_c(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o, \tilde{t}) = f_c(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o) = n - \sum_{j=1}^{n} \bar{t}_j = n - \sum_{j=1}^{n} \tilde{t}_j.$$

It follows that also $(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o, \tilde{t})$ is best weight in Λ_c for DMU_o, and $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ is an optimal solution of *Model (5)*.

Theorem 6. Let $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o, \bar{t})$ be an optimal solution of the following model:

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} t_j, \\ \text{s.t.} \quad & v^t x_o + v_o - (u^t y_o + u_o) = 0, \\ & v^t x_j + v_o - (u^t y_j + u_o) + M t_j \geq 0, \ \ \text{for all} \quad j \ \ (M \gg 0), \\ & u \geq 1, \ \ v \geq 1, \ u_o \geq 0, \ v_o \geq 0, \ \ t_j \epsilon\{0,1\}. \end{aligned}$$

Then $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ is best weight in input-oriented (output-oriented) in Λ_c for DMU_o.

Proof: similar to the proof of *Theorem 5*.

Corollary 10. Let $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ be an optimal solution of *Model (6)*, then strictly Λ_v-efficiency of

$$DMU_o = \frac{f_v(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)}{n} = \frac{n - \sum_{j=1}^{n} \bar{t}_j}{n}.$$

Proof: *Theorem 6*.

Corollary 11. DMU_o is strictly Λ_v-efficient if only if the optimal objective function value of *Model (6)* be zero.

Proof: *Theorem 6*.

Theorem 7. Let $(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o, \tilde{t})$ be an optimal solution for the following model:

$$\min \sum_{j\neq o} t_j,$$
$$\text{s.t.}\quad \begin{aligned} & v^t x_o + v_o - (u^t y_o + u_o) = 0\,, v^t x_o \geq \varepsilon\ (u^t y_o \geq \varepsilon),\\ & v^t x_j + v_o - (u^t y_j + u_o) + M t_j \geq \varepsilon,\ \ j \neq o\ (\,M \gg 0),\\ & u \geq 0,\ \ v \geq 0,\ t_j \epsilon \{0,1\}, j \neq o,. \end{aligned}$$

where $\tilde{t} = (\tilde{t}_1, \dots, \tilde{t}_{o-1}, \tilde{t}_{o+1}, \dots, \tilde{t}_n)$. Then if $\sum_{j\neq o} \tilde{t}_j < n-1$, $(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o)$ is a relatively strongest weight in input-oriented (output-oriented) in Λ_c for DMU_o. But if $\sum_{j\neq o} \tilde{t}_j = n-1$, there is not any relatively strongest weight in input-oriented (output-oriented)in Λ_c for DMU_o. Conversely if there is not any relatively strongest weight in input-oriented (output-oriented) in Λ_c for DMU_o, then the optimal objective function value of *Model (7)* is equal to $n-1$. But if there exists a $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o) \epsilon \Lambda_v$ such that $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ is relatively strongest weight in input-oriented (output-oriented) in Λ_v for DMU_o, then by taking $p = g_v(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$,

$$\left\{DMU_{j_1}, \dots, DMU_{j_p}\right\} = \left\{DMU_j \middle| j \epsilon \{1, \dots, n\}\ \&\ \bar{v}^t x_j + \bar{v}_o \geq \bar{u}^t y_j + \bar{u}_o\right\}. \tag{19}$$

And

$$\bar{t} = (\bar{t}_1, \dots, \bar{t}_{o-1}, \bar{t}_{o+1}, \dots, \bar{t}_n).$$

With

$$\bar{t}_j = \begin{cases} 0, & j \epsilon \{j_1, \dots, j_p\},\\ 1, & j \epsilon \{1, \dots, n\} - \{j_1, \dots, j_p, o\}. \end{cases} \tag{20}$$

$(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o, \bar{t})$ is an optimal solution for *Model (8)* and $\sum_{j\neq o} \bar{t}_j < n-1$.

Proof: if there is a $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o) \epsilon \Lambda_v$ such that $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ is relatively strongest weight in input-oriented (output-oriented) in Λ_c for DMU_o, then $g_v(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o) \geq 1$ and $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o, \bar{t})$, where $\bar{t}$ is defined by *Models (11)* and *(12)*, is a feasible solution for *Model (7)*. Thus, since $(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o, \tilde{t})$ is an optimal solution for *Model (7)*, we have$\sum_{j\neq o} \tilde{t}_j \leq \sum_{j\neq o} \bar{t}_j$. Therefore,

$$g_v(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o) \geq g_v(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o) \geq 1.$$

Thus the optimal objective function value of *Model (7)* is less $n-1$. On the other hand, $g_c(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o) \leq g_c(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$, since $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ is relatively strongest weight in Λ_c for DMU_o. Consequently, $g_c(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o) = g_c(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$. Thus $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o, \bar{t})$ is an optimal solution for *Model (7)*, and $(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o)$ is a relatively strongest weight in input-oriented (output-oriented in Λ_c for DMU_o. Also it easy to show, if there is not any relatively strongest weight in input-oriented (output-oriented in Λ_c for DMU_o, then the optimal objective function value of *Model (7)* is equal to $n-1$.

Theorem 8. Let $(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o, \tilde{t})$ be an optimal solution for the following model,

$$\min \sum_{j\neq o} t_j,$$
$$\text{s.t.}\quad \begin{aligned} & v^t x_o + v_o - (u^t y_o + u_o) = 0\,, v^t x_o \geq 1\ (u^t y_o \geq 1),\\ & v^t x_j + v_o - (u^t y_j + u_o) + M t_j \geq 1,\ \ j \neq o\ \ (\,M \gg 0),\\ & u \geq 0,\ \ v \geq 0,\ t_j \epsilon \{0,1\}, j \neq o, \end{aligned}$$

where $\tilde{t} = (\tilde{t}_1, \dots, \tilde{t}_{o-1}, \tilde{t}_{o+1}, \dots, \tilde{t}_n)$. Then if $\sum_{j\neq o} \tilde{t}_j < n-1$, $(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o)$ is a relatively strongest weight in input-oriented (output-oriented) in Λ_c for DMU_o. But if $\sum_{j\neq o} \tilde{t}_j = n-1$, there is not any relatively strongest weight in input-oriented (output-oriented)in Λ_c for DMU_o.

Proof: similar to the proof of *Theorem 7*.

Corollary 12. let $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ be an optimal solution of *Model (8)*, then

input-oriented (output-oriented) Λ_v-efficiency of,

$$\mathrm{DMU_o} = \frac{g_v(\bar{u},\bar{v},\bar{u}_o,\bar{v}_o)}{n-1} = \frac{(n-1)-\sum_{j\neq o} t_j}{n-1}.$$

Proof: *Theorem 8*.

Corollary 13. $\mathrm{DMU_o}$ is input-oriented (output-oriented) Λ_v-efficient if only if the optimal objective function value of *Model (6)* is zero.

Proof: *Theorem 8*.

Theorem 9. Let $(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o, \tilde{t})$ be an optimal solution for the following model:

$$\begin{aligned} \min \quad & \sum_{j\neq o} t_j, \\ \text{s.t.} \quad & v^t x_o + v_o - (u^t y_o + u_o) = 0\,, v^t x_o \geq \varepsilon\,, u^t y_o \geq \varepsilon, \\ & v^t x_j + v_o - (u^t y_j + u_o) + M t_j \geq \varepsilon, \;\; j \neq o \; (M \gg 0), \\ & u \geq 0, \;\; v \geq 0, \; t_j \epsilon \{0,1\}, j \neq o, \end{aligned}$$

where $\tilde{t} = (\tilde{t}_1, \dots, \tilde{t}_{o-1}, \tilde{t}_{o+1}, \dots, \tilde{t}_n)$. Then if $\sum_{j\neq o} \tilde{t}_j < n-1$, $(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o)$ is a relatively strongest weight in Λ_c for $\mathrm{DMU_o}$. But if $\sum_{j\neq o} \tilde{t}_j = n-1$, there is not any relatively strongest weight in Λ_c for $\mathrm{DMU_o}$. Conversely if there is not any relatively strongest weight in Λ_c for $\mathrm{DMU_o}$, then the optimal objective function value of *Model (9)* is equal to $n-1$. But if there exists a $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o) \epsilon \Lambda_v$ such that $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ is relatively strongest weight in Λ_v for $\mathrm{DMU_o}$, then $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o, \bar{t})$, where $\bar{t}$ is defined by *Models (10)* and *(11)*, is an optimal solution for *Model (9)* and $\sum_{j\neq o} \bar{t}_j < n-1$.

Proof: similar to the proof of *Theorem 7*.

Theorem 10. Let $(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o, \tilde{t})$ be an optimal solution for the following model,

$$\begin{aligned} \min \quad & \sum_{j\neq o} t_j, \\ \text{s.t.} \quad & v^t x_o + v_o - (u^t y_o + u_o) = 0\,, v^t x_o \geq 1\,, u^t y_o \geq 1, \\ & v^t x_j + v_o - (u^t y_j + u_o) + M t_j \geq 1, \;\; j \neq o \; (M \gg 0), \\ & u \geq 0, \;\; v \geq 0, \; t_j \epsilon \{0,1\}, j \neq o. \end{aligned}$$

where $\tilde{t} = (\tilde{t}_1, \dots, \tilde{t}_{o-1}, \tilde{t}_{o+1}, \dots, \tilde{t}_n)$. Then if $\sum_{j\neq o} \tilde{t}_j < n-1$, $(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o)$ is a relatively strongest weight in Λ_c for $\mathrm{DMU_o}$. But if $\sum_{j\neq o} \tilde{t}_j = n-1$, there is not any relatively strongest weight in Λ_c for $\mathrm{DMU_o}$.

Proof: similar to the proof of *Theorem 7*.

Corollary 14. let $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ be an optimal solution of *Model (10)*, then

strongly Λ_v-efficiency of,

$$\mathrm{DMU_o} = \frac{g_v(\bar{u},\bar{v},\bar{u}_o,\bar{v}_o)}{n-1} = \frac{(n-1)-\sum_{j\neq o} t_j}{n-1}.$$

Proof: *Theorem 10*.

Corollary 14. $\mathrm{DMU_o}$ is strongly Λ_v-efficient if only if the optimal objective function value of *Model (10)* be zero.

Theorem 11. Let $(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o, \tilde{t})$ be an optimal solution for the following model:

$$\min \sum_{j\neq o} t_j,$$
$$\text{s.t.}\quad \begin{array}{l} v^t x_o + v_o - (u^t y_o + u_o) = 0, \\ v^t x_j + v_o - (u^t y_j + u_o) + M t_j \geq \varepsilon,\ \ j \neq o\ (M \gg 0), \\ u \geq 1\varepsilon,\ \ v \geq 1\varepsilon,\ t_j \epsilon \{0,1\}, j \neq o, \end{array}$$

where $\tilde{t} = (\tilde{t}_1, \dots, \tilde{t}_{o-1}, \tilde{t}_{o+1}, \dots, \tilde{t}_n)$. Then if $\sum_{j\neq o} \tilde{t}_j < n-1$, $(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o)$ is a strongest weight) in Λ_c for DMU_o. But if $\sum_{j\neq o} \tilde{t}_j = n-1$, there is not any in Λ_c for DMU_o. Conversely if there is not any strongest weight in Λ_c for DMU_o, then the optimal objective function value of *Model (11)* is equal to $n-1$. But if there exists a $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o) \epsilon \Lambda_v$ such that $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o)$ is strongest weight in Λ_v for DMU_o, then $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o, \bar{t})$, where $\bar{t}$ is defined by *Models (11)* and *(12)*, is an optimal solution for *Model (11)* and $\sum_{j\neq o} \bar{t}_j < n-1$.

Proof: similar to the proof of *Theorem 7*.

Theorem 12. Let $(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o, \tilde{t})$ be an optimal solution for the following model,

$$\min \sum_{j\neq o} t_j,$$
$$\text{s.t.}\quad \begin{array}{l} v^t x_o + v_o - (u^t y_o + u_o) = 0, \\ v^t x_j + v_o - (u^t y_j + u_o) + M t_j \geq 1,\ \ j \neq o\ \ (M \gg 0), \\ u \geq 1,\ \ v \geq 1,\ t_j \epsilon \{0,1\}, j \neq o, \end{array}$$

where $\tilde{t} = (\tilde{t}_1, \dots, \tilde{t}_{o-1}, \tilde{t}_{o+1}, \dots, \tilde{t}_n)$. Then if $\sum_{j\neq o} \tilde{t}_j < n-1$, $(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o)$ is strongest weight in Λ_c for DMU_o. But if $\sum_{j\neq o} \tilde{t}_j = n-1$, there is not any strongest weight in Λ_c for DMU_o.

Proo: similar to the proof of *Theorem 7*.

Theorem 13. DMU_o is input oriented (output oriented) Λ_c-efficient if only if DMU_o is input oriented BCC efficient.

Proof: let DMU_o be input oriented Λ_c-efficient, then, by *Corollary 2*, the optimal objective function value of *Model (5)* is zero. Thus letting $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o, \bar{t})$ be an optimal solution for *Model (5)*, we have

$$\bar{u} \geq 0, \bar{v} \geq 0,\ \ \bar{v}^t x_o + \bar{v}_o = \bar{u}^t y_o + \bar{u}_o,\ \bar{v}^t x_o = 1 (\bar{u}^t y_o = 1),\ \ \bar{v}^t x_j + \bar{v}_o - (\bar{u}^t y_j + \bar{u}_o) \geq 0\ ,\ \ j = 1, .., n.$$

Hence $(\bar{u}, \bar{v}, \overline{w}_o)$, where $\overline{w}_o = \bar{u}_o - \bar{v}_o$, is an optimal solution for *Model (2)*. therefore DMU_o is input oriented (output oriented) BCC - efficient. Conversely, let DMU_o is input oriented (output oriented) BCC - efficient, and let $(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o)$ is an optimal solution of *Model (2)*, then

$$\tilde{u} \geq 0, \tilde{v} \geq 0, \qquad \tilde{v}^t x_o = \tilde{u}^t y_o + \widetilde{w}_o, \tilde{v}^t x_o = 1 (\tilde{u}^t y_o = 1), \qquad \tilde{v}^t x_j - \left(\tilde{u}^t y_j + \widetilde{w}_o\right) \geq 0\ ,\ \ j = 1, .., n,$$

Thus $(\tilde{u}, \tilde{v}, \tilde{u}_o, \tilde{v}_o, \tilde{t})$, where

$$\bar{u}_o = \begin{cases} \widetilde{w}_o & \widetilde{w}_o \geq 0, \\ 0 & \widetilde{w}_o < 0, \end{cases}$$

$$\bar{v}_o = \begin{cases} 0 & \widetilde{w}_o \geq 0, \\ -\widetilde{w}_o & \widetilde{w}_o < 0. \end{cases}$$

and $\bar{t} = 0 \epsilon \mathbb{R}^n$, is a feasible solution for *Model (2)*. Therefore the optimal objective function value of *Model (5)* is zero. Hence, by *Corollary 2*, DMU_o is input oriented (output oriented) Λ_c-efficient.

Theorem 14. If $\mathrm{DMU_o}$ be Λ_c- efficient, then $\mathrm{DMU_o}$ is both input oriented BCC - efficient. and output oriented BCC – efficient.

Proof: let $\mathrm{DMU_o}$ is Λ_c- efficient. Then, by *Corollary 6*, the optimal objective function value *Model (7)* is zero, thus letting $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o, \bar{t})$ be an optimal solution for the model, we have

$$\bar{u} \geq 0, \bar{v} \geq 0,\ \bar{v}^t x_o + \bar{v}_o = \bar{u}^t y_o + \bar{u}_o,\ \bar{v}^t x_o \geq 1,\ \bar{u}^t y_o \geq 1,\ \bar{v}^t x_j + \bar{v}_o - (\bar{u}^t y_j + \bar{u}_o) \geq 0\ ,\ \ j = 1, .., n.$$

So that by taking $k = \bar{v}^t x_o$, $\tilde{u} = (\bar{u}/k)$,and $\tilde{v} = (\bar{v}/k)$, we have

$$\tilde{u} \geq 0, \tilde{v} \geq 0,\ \tilde{v}^t x_o + \tilde{v}_o = \tilde{u}^t y_o + \tilde{u}_o,\ \tilde{v}^t x_o = 1,\ \tilde{v}^t x_j + \tilde{v}_o - (\tilde{u}^t y_j + \tilde{u}_o) \geq 0\ ,\ \ j = 1, .., n.$$

Thus $(\tilde{u}, \tilde{v}, \tilde{w}_o)$, where $\tilde{w}_o = \tilde{u}_o - \tilde{v}_o$, is a feasible solution for *Model (4)*, and since $\tilde{v}^t x_o = 1$ it follows that the optimal objective function value of *Model (4)* is equal to one. Hence $\mathrm{DMU_o}$ is input oriented BCC - efficient. Similarly, we can show that $\mathrm{DMU_o}$ is output oriented BCC – efficient.

Theorem 15. $\mathrm{DMU_o}$ is strictly Λ_v- efficient if only if $\mathrm{DMU_o}$ is BCC - efficient.

Proof: let $\mathrm{DMU_o}$ is strictly Λ_v- efficient. Then, by *Corollary 6*, the optimal objective function value *Model (7)* is zero, thus letting $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o, \bar{t})$ be an optimal solution for the model, we have

$$\bar{u} \geq 1, \bar{v} \geq 1,\ \bar{v}^t x_o + \bar{v}_o = \bar{u}^t y_o + \bar{u}_o,\ \bar{v}^t x_j + \bar{v}_o - (\bar{u}^t y_j + \bar{u}_o) \geq 0\ ,\ \ j = 1, .., n.$$

Thus $(\bar{u}, \bar{v}, \bar{w}_o)$, where $\bar{w}_o = \bar{u}_o - \bar{v}_o$, is a feasible solution for the following model

$$\begin{aligned} \min \quad & v^t x_o - (u^t y_o + \bar{w}_o), \\ \text{s.t.} \quad & v^t x_j - (u^t y_j + \bar{w}_o) \geq 0 \ \text{ for all } j, \\ & u \geq 1,\ \ v \geq 1, \end{aligned}$$

which for the feasible solution the objective function value of the model is equal to zero. Therefore the optimal objective function value *Model (9)* is zero. Therefore, by strong duality theorem, the optimal objective function value of the following model,

$$\begin{aligned} \max \quad & \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+, \\ \text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{io} \ \text{ for all } i, \\ & \sum_{j=1}^{n} \lambda_j y_{ij} - s_r^+ = y_{ro} \text{ for all } r, \\ & \lambda_j \geq 0, s_i^- \geq 0, s_r^+ \geq 0 \ \text{ for all } j, \text{for all } i, \text{for all } r, \end{aligned}$$

which is dual form of *Model (9)*, equal to one. Consequently $\mathrm{DMU_o}$ is BCC – efficient. It is easy to show that if $\mathrm{DMU_o}$ is BCC – efficient, then $\mathrm{DMU_o}$ is strictly Λ_v- efficient.

Theorem 16. $\mathrm{DMU_o}$ is input oriented strongly Λ_c- efficient if only if $\mathrm{DMU_o}$ is extreme BCC - efficient.

Proof: let $\mathrm{DMU_o}$ be input oriented strongly Λ_c- efficient. Then, by Corollary 3.8, the optimal objective function value *Model (8)* is zero, thus letting $(\bar{u}, \bar{v}, \bar{u}_o, \bar{v}_o, \bar{t})$ be an optimal solution for the model, where

$$\bar{t} = (\bar{t}_1, \dots, \bar{t}_{o-1}, \bar{t}_{o+1}, \dots, \bar{t}_n) = 0 \in \mathbb{R}^{n-1}.$$

we have

$\bar{u} \geq 0, \bar{v} \geq 0,\ \bar{u}_o \geq 0,\ \bar{v}_o \geq 0,\ \bar{v}^t x_o \geq 1,\ \bar{v}^t x_o + \bar{v}_o = \bar{u}^t y_o + \bar{u}_o,\ \bar{v}^t x_j + \bar{v}_o - (\bar{u}^t y_j + \bar{u}_o) \geq 1\ ,\ \ j \neq o.$

Thus $(\bar{u}, \bar{v}, \bar{w}_o)$, where $\bar{w}_o = \bar{u}_o - \bar{v}_o$, is a feasible solution for the following model

$$\begin{aligned} \min \quad & v^t x_o - (u^t y_o + w_o), \\ \text{s.t.} \quad & v^t x_o - (u^t y_o + w_o) \geq 0, \\ & v^t x_j - \left(u^t y_j + w_o\right) \geq 1 \ \ j \neq o, \\ & v^t x_o \geq 1, \\ & u \geq 0,\ \ v \geq 0, w_o \text{is free.} \end{aligned}$$

Therefore, by strong duality theorem, the optimal objective function value of the following model

$$\begin{aligned} \max \quad & \theta + \varepsilon \sum_{j \neq o} \lambda_j, \\ \text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_j + \theta x_o \leq x_{o,} \\ & \sum_{j=1}^{n} \lambda_j y_j \geq y_{o,} \\ & \sum_{j=1}^{n} \lambda_j = 1, \\ & \lambda_j \geq 0,\ \ \text{for all}\ \ j, \end{aligned}$$

which is dual form of *Model (9)*, equal to one. Hence, by *Theorem 2*, DMU_o is extreme BCC-efficient. Conversely, if DMU_o is extreme BCC-efficient, by *Theorem 2* and *Corollary 8*, DMU_o is input oriented strongly Λ_c- efficient.

4 | Summary and Conclusion

In Section 3, we provided models to obtain non negative weights for inputs, nonnegative weights for outputs, a nonnegative scalar corresponding to inputs and a nonnegative scalar corresponding to outputs which for the weights and scalars, the number of which DMUs for each one its virtual output in addition to the scalar corresponding to inputs does not exceed (is less than, if any) its virtual input in addition to the scalar corresponding to inputs be maximum, provided that for DMU under evaluation, the virtual output in addition to the scalar corresponding to inputs does not exceed (is less than, if any) the virtual input in addition to the scalar corresponding to inputs and the virtual input will be positive. We called these weights and scalars, the relatively best weight in input-oriented (the relatively strongest weight in input-oriented, if any) for the DMU under evaluation, and if all of the weights be positive we called them best weight in input-oriented (the strongest weight in input-oriented, if any) for the DMU under evaluation. The relatively best weight in input-oriented (the relatively strongest weight in input-oriented) indicates normal vector of a superface in the PPS with VRS assumption that the DMU under evaluation is on the superface and the maximum number of which DMUs their performance are no worse than (is better than) the DMU under evaluation separate from the rest of DMUs, with the constraint that the virtual input be positive. Accordingly, we can interpret the rest of the definitions of non-negative weights for inputs and for outputs and nonnegative scalars related to inputs and outputs. Also in this paper, we presented the relationship between these definitions of efficiency with efficiency in the DEA models with VRS assumption. These normal vectors can be applied as a criterion for efficiency analysis and ranking of a set of peer DMUs with interval scale data. Specially, the relatively strongest weight in input-oriented (in output-oriented), both indicate extreme CCR-efficiency and provide a performance measure DMU_o with interval scale inputs and/or outputs. Also the relatively strongest weight and the strongest weight can be applied for ranking extreme CCR-efficient DMUs and BCC-inefficient DMUs.

Conflict of Interest

The authors declare that they have no conflict of interest regarding the publication of this paper.

Data Availability

No datasets were generated or analyzed during the current study. All necessary theoretical models and proofs are included within the article.

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