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Non-Radial Piecewise Linear DEA Model

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Abstract

In standard Data Envelopment Analysis (DEA) models it is as- sumed that the aggregate output (input) is a pure linear function of each output (input). But in real life situations linear pricing may not sufficiently reveal the differences in value which are created from one Decision Making Unit (DMU) to another. Thus for overcoming this difficulty a generalization of the? DEA methodology has been presented that incorporates piece- wise linear functions of factors. In this paper, considering the benefits of nonradial DEA models over that of radial ones, this subject has been expanded and new model have been presented. Also considering this situation the issue of efficiency assessment, finding targets and identifying reference set in presence of trade off technology has been discussed. Furthermore, the above-mentioned mode is compared to those obtained through radial ones and an example is provided for the sake of lucidity.

Keywords: Data envelopment analysis, Piecewise linear function, Trade offs, Target, Marginal value, Nonradial modes.

1 | Introduction

Data Envelopment Analysis (DEA) developed by Charnes et al. [1] and generalized later has become one of the most widely used methods in oper- ations research and management science. DEA is a task oriented approach and focuses on an important task to evaluate relative (technical) efficiency of comparable Decision Making Units (DMUs) essentially performing the same task and this is a reason for tits success [2]–[5]. Based on information about existing data on the performance of the units and some preliminary assumptions, the purpose of DEA, based on the set of available DMUs and to project all DMUs on to this frontier, is to

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empirically characterize the socalled efficient frontier. If a DMU lies on the frontier, it is referred to as an efficient unit, otherwise inefficient.

Benchmarking is the process of identifying the highest standards of excellence for products and therefore making the necessary improvements to acquire those standards, which are referred to as best practices [6]. Trough solving DEA models benchmark units corresponds to each inefficient unit canbe identified.

In DEA, a pair of dual linear programming problems are mentioned as envelopment and multiplier models. Additional weight restrictions can be imposed on multiplier DEA model in different cases, which are mainly based on managerial significance of inputs and outputs or input costs and output prices. Using weight restrictions offers real practical advantages. The information about production trade-offs between inputs and outputs can be incorporated into the DEA models. Podinovski [7] has suggested two ways for treating production trade-offs. One way is considering trade-offs as ad- ditional terms in modifying composite DMUs and the other is incorporating weight restrictions in multiplier model. Once trade-offs have been converted to the weight restrictions in multiplier model, an explicit understanding of the trade-off technology comes thereby. Podinovski [8] proposed the trade- off approach for constructing weight restrictions. While considering trade-off technology, Production Possibility Set expands. The important advantage of this method is that the efficiency score still has its traditional meaning as the highest radial improvement factor, and radial targets for inefficient units can be achieved.

Cook et al. [9] examined the efficiency of maintenance patrols in the province of Ontario, Canada. More recently [10], they modified the original model by the concept of output erosion under decreasing inputs has been addressed. An other practical model that has been provided by Cook et al. [11] considers a type of variable which has nonlinear impact on efficiency. In such circumstances either a nonincreasing or nondecreasing set of multipliers for larger magnitude of factors describes the weight func- tion. These variables are respectively called exhibiting Diminishing Marginal Value (DMV) and exhibiting Increasing Marginal Value (IMV). The imposition of weight restrictions is one of the significant factors when applying piecewise linear DEA model. By introducing Piecewise Linear model, they have demonstrated that the traditional linear structure can not reflect real situations.

In this paper,we restrict the analysis to the nonradial model, where simoultaniously inputs are contracted and outputs are increased. We show that the proposed nonradial piecewise linear DEA model can reveal real sit- uations and moreover it is impossible to have the Pareto efficient targets. Also with an example we will demonstrate how this method works.

The paper unfolds as follows. In Section 2, piecewise linear DEA model will be briefly reviewed. In Section 3, the proposed method based on MIP model will be presented. An illustrative example is documented in Section 4 and Section 5 concludes the paper.

2 | Piecewise Linear DEA Model

DEA has become a widespread analytical tool for evaluating the relative ef- ficiency of comparable firms. A fundamental assumption behind the DEA technique is that if a given DMU is producing y units of output with x units of input then, other resemble DMUs should also be able to do the same if they were to operate efficiently.

Let DMU_0 denote a unit from a total n units which relative efficiency is being evaluated. Define $x_0 \in R_+^m$ and $y_0 \in R_+^s$ as inputs and outputs of DMU_0 .

The most general way to characterize production technology is production possibility set T, which is defined with a set of semipositive (x, y) as

$$T = \{(x, y) \mid x \geq \sum_{j=1}^{n} \lambda_j x_j, \quad y \leq \sum_{j=1}^{n} \lambda_j y_j, \quad \lambda_j \geq 0, \quad j = 1, \ldots, n\}.$$

The constant returns to scale form of the enveloping problem which was first introduced by Charnes et al. [1], is as follows:

$$\begin{aligned}
&\min\ \theta\\
&\text{s.t.}\ \sum_{j=1}^{n}\lambda_j x_{ij} \le \theta x_{io},\ i=1,\dots,m,\\
&\qquad \sum_{j=1}^{n}\lambda_j y_{rj} \ge y_{ro},\quad r=1,\dots,s,\\
&\qquad \lambda_j \ge 0,\ j=1,\dots,n.
\end{aligned} \tag{1}$$

The dual problem for objective function and *Constraint (1)*, is to solve for vectors $u \in R^s$ and $v \in R^m$ such that:

$$\begin{aligned}
&\max\ \sum_{r=1}^{s}u_r y_{ro},\\
&\text{s.t.}\ \sum_{i=1}^{m}u_i x_{io} = 1,\\
&\qquad \sum_{r=1}^{s}u_r y_{ri} - \sum_{i=1}^{m}u_i x_{ij} \le 0,\ j=1,\dots,n,\\
&\qquad u \ge 0,\ u \ge 0.
\end{aligned} \tag{2}$$

In the following model, the aggregate output has been deemed as a linear function of each output. Cook et al. [11] have provided that linear pricing may not adequately reflect real situations. Therefore they proposed a model in which a nonincreasing or nondecreasing set of multipliers for larger magnitude of the factors describes the weight function. These variables are respectively called exhibiting DMV and exhibiting IMV. In presence of DMV variable, which has nonlinear impact on efficiency, from the theory of piecewise linear programming, the scale can be divided up into k segments and each variable in these segments can be assumed to behave linearly. With this logic, the scale of variable which indicates DMV behavior, should view as consisting of k_r ranges $[0, L_1], (L_1, L_2], \dots, (L_{kr-1}, L_{kr}]$. let u_{r_k} be the value which is given to the portion of y_{rj} that lies in the k^{th} range. If $y_{rj} \in (L_{k_j-1}, L_{k_j]}$ then the parameters y_{rj}^k are defined as follows:

$$y_{rj}^k = \begin{cases} L_k, & \text{if} \quad k=1,\\ L_k - L_{k-1}, & \text{if} \quad k=2,\dots,k_j-1,\\ y_{rj} - L_{k-1}, & \text{if} \quad k=k,\\ 0, & \text{if} \quad k>k_j. \end{cases} \tag{3}$$

The piecewise linear DEA model which has been proposed by Cook et al. [11] is as follows:

$$
\begin{aligned}
&\max \sum_{r\in R_1} u_r y_{ro} + \sum_{r\in R_2}\sum_{k=1}^{K_r} u_{rk} y_{ro}^k,\\
&\text{s.t. } \sum_{i=1}^{m} U_i x_{io} = 1,\\
&\sum_{r\in R_1} u_r y_{rj} + \sum_{r\in R_2}\sum_{k=1}^{K_r} u_{r_k} y_{rj}^k - \sum_{i=1}^{m} U_i x_{ij} \le 0 \ , \ \text{for all } j\in J\\
&u_{r_{k+1}} a_{r_k} \le u_{r_k} \le u_{r_{k+1}} b_{r_k}, \quad k=1,\dots,K_r, r\in R_2 \qquad \text{(a)}\\
&u_{r_1} a_{r_1 r_2} y_{r_2 j} \le \sum_{k=1}^{K_{r_2}} u_{r_{2k}} y_{r_2 j}^k \le u_{r_1} b_{r_1 r_2} y_{r_2 j}, \qquad j\in J \qquad \text{(b)}\\
&y_{r_k} \ge \varepsilon \quad r\in R_2,\ k=1,\dots,k_r,\\
&u_r \ge \varepsilon \quad r\in R_1,\quad u\ge 0.
\end{aligned} \tag{4}
$$

In the above model R_1 and R_2 respectively, are used to denote sets of regular and DMV outputs , and DMV outputs, $J=\{1,\dots,n\}$ and $r_1\in R_1$, $r_2\in R_2$. The linear equivalent of piecewise linear function $\sum_{k=1}^{K} u_{r_k} y_{rj}^k$, is given by $f(y_{rj)=u_r(j)y_{rj}},$ where $u_r(j)$ is a convex combination of $\{u_{r_k}\}_{k=1}^{K}$ [11]. Therefore, each of the *Constrains* in *(b)*, which is a comparison of a set of multipli ers for a variable that exhibits DMV to another variable for which this is not true, creates a form of Generalized Assurance Region (GAR) in context of [12]. Also, The above model forms the Cone Ratio DEA structure, by imposing a set of linear constrains on output multipliers as mentioned in [13]. An emphasis is placed on utilizing proportional weight bounds, as is necessary that $\{u_{r_k}\}_{k=1}^{k_r}$ should form a decreasing sequence. Thus, the *Constraint (a)* is imposed to capture the idea of such a sequence.

It is noteworthy that a_{r_k} and b_{r_k} would take on values strictly greater than one for the DMV variable. The parameters a_{r_1,r_2} and b_{r_1,r_2} are the lower and upper bounds on the ratios of pairs of regular and DMV variables. The choice of number, width of ranges and bounds on the ratios of pairs of variables, would need to be carefully determined by an analyst.

In such circumstances, the constant returns to scale form of the enveloping problem is as follows:

$$
\begin{aligned}
&\min \theta\\
&\text{s.t. } \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta x_{io}, \quad i=,\dots,m,\\
&\quad \sum_{j=1}^{n} \lambda_j y_{rj} + \sum_{t=1}^{l} \pi_t Q_{rt} \ge y_{ro}, \quad r\in R_1,\\
&\quad \sum_{j=1}^{n} \lambda_i y_{rj}^k + \sum_{t=1}^{l} \pi_t Q_{rt}^k \ge y_{ro}^k, \quad r\in R_2\ ,\ k=1,\dots,k_r,\\
&\quad \lambda_j \ge 0\ ,\ \pi_t \ge 0\ ,\ j=1,\dots,n,\ \ t=1,\dots,l.
\end{aligned} \tag{5}
$$

$Q_{rt}, r\in R_1,\ t=1,\dots,l.$ are the coefficient vectors of imposed weight restric- tions on the r[th] output weights in multiplier model. Also, $Q_{rt}^k r\in R_2, k=1,\dots,k_{rt}, t=1,\dots,l$ are the coefficient vectors of imposed weight restrictions on k_r portions of the r[th] output weights. On basis of what Podinovski [7] has provided, the above envelopment model reveals the incorporation of production tradeoffs into the analysis. As is shown by Podinovski [8], the dual relationship reveals that the inclusion of tradeoffs into the envelopment model is the same as an inclusion of weight restrictions into the multiplier form. In *Model (5)* production tradeoffs between outputs are incorporated.

3 | Nonradial Piecewise Linear DEA Model

In this section nonradial piecewise linear DEA model will be presented. In this model both IMV and DMV variables are incorporated. The obtained scaler measure through solving this model deals directly with the regular and IMV input excesses and regular and DMV output shortfalls for entities. It will be discussed that this measure will be unit invariant and monotone and also that this measure is determined only by consulting the reference of the DMU.

According to what has been mooted by Cook and Zhu [11], input oriented PLCCR model has been presented in order to efficiency evaluation in presence of an outputs which have nonlinear impact on efficiency. It should be noted that the proposed input oriented model has not the capability of eval- uating efficiency in presence of inputs which have nonlinear behavior. The same is true for output oriented model in presence of outputs which have nonlinear behavior. In such circumstances the necessity of having models which have the ability to incorporate with inputs and outputs which have nonlinear impact on efficiency is being felt. Therefore we come up to introduce a model which has the ability of efficiency assessment in presence of inputs and outputs which have nonlinear behavior where a nonincreasing or nondecreasing set of multipliers for larger magnitude of the factors describes the weight function. These variables are respectively referred to as exhibiting DMV and exhibiting IMV. In presence of a variable, which has nonlinear impact on efficiency, from the theory of piecewise linear programming the scale can be divided up into k segments and each variable in these segments can be as sumed to behave linearly. With this logic, the scale of variable which exhibits as having nonlinear behavior, should view as consisting of k_r ranges $[0, L_1]$, $(L_1, L_2], \ldots, (L_{k_r-1}, L_{k_r}]$. The choice of number, width of ranges and bounds on the ratios of pairs of variables, would need to be carefully determined by an analyst.

Let u_{r_k} be the value which is given to the portion of y_{rj} that lies in the k^{th} range.

If $y_{rj} \in (L_{k_j-1}, L_{k_j}]$ then the parameters y_{rj}^k are defined as follows:

$$y_{rj}^k = \begin{cases} L_k, & \text{if} \quad k = 1, \\ L_k - L_{k-1}, & \text{if} \quad k = 2, \ldots, k_j - 1, \\ y_{rj} - L_{k-1}, & \text{if} \quad k = k_j, \\ 0, & \text{if} \quad k > k_j. \end{cases} \tag{6}$$

An emphases is placed on utilizing proportional weight bounds for it is neccessary that $\{u_{r_k}\}_{k=1}^{k_r}$ should form a decreasing sequence. It is noteworthy that these weight bounds would take on values strictly greater than one for the DMV variable.

Let v_{ik} be the value which is given to the portion of x_{ij} that lies in the k^{th} range.

If $x_{ij} \in (L_{k_j-1}, L_{k_j}]$ then the parameters x_{ij}^k are defined as follows:

$$x_{ij}^k = \begin{cases} L_k, & \text{if} \quad k = 1, \\ L_k - L_{k-1}, & \text{if} \quad k = 2, \ldots, k_j - 1, \\ x_{ij} - L_{k-1}, & \text{if} \quad k = k_{j,} \\ 0, & \text{if} \quad k > k_j. \end{cases} \tag{7}$$

As it is noted it is necessary that $\{u_{i_k}\}_{k=1}^{k_i}$ should form a increasing sequence.

It is noteworthy that these weight bounds would take on values strictly less than one for the IMV variable.

Let I_2 and R_2 be the sets of inputs and outputs which respectively have IMV and DMV behavior and I_1 and R_1 be the sets of regular inputs and outputs. As is shown by Podinovski [8], the dual relationship reveals that the inclusion of tradeoffs into the envelopment model is the same as an inclusion of weight restrictions into the multiplier form. On basis of what Podinovski [7] has provided, the following envelopment model reveals the incorporation of production tradeoffs into the analysis, where $Q^k_{rt}; r \in R_2, k = 1,...,k_r, t = 1,...,l$ are the coefficient vectors of imposed weight restrictions on k_r portions of the r^th^ output weights and P^k_{it} ; $i \in I_2$, $k = 1,...,k_i, t = 1,...,l$ are the coefficient vectors of imposed weight restrictions on k_r portions of the ith input weights.

$s^-_i, i \in I_1$ and $s^{-k}_i, i \in I_2, k = 1,...,k_i$ are the slacks of regular and IMV inputs respectively. $s^+_t, r \in R_1$ and $s^{+k}_r, r \in R_2, k = 1,...,k_r$ are the slacks of regular and DMV outputs respectively.

Consider

$$y^{-k}_{rj} = \begin{cases} 0, & \text{if} \quad k = 1, \\ 0, & \text{if} \quad k = 2,...,k_j - 1, \\ L_k - y_{rj}, & \text{if} \quad k = k_j, \\ L_k - L_{k-1}, & \text{if} \quad k > k_j. \end{cases} \tag{8}$$

In the following model, let $v_0 = 0$ and M be a positive constant " the big M". It is noteworthy to say that, s^{+k}_r is forced to zero by the binary variable v. Clearly, when the lower ranges have not been filled, selecting $v = 1$ forces the slack , s^{+k}_r , to zero. $k_r, r \in R_2$ represents number of intervals, $(L_{k-1}, L_k]$, that defining the piecewise linear function for DMV output r. The bundles of *Constraints (a)-(c)* are imposed into the model to confine the nonradial improvements in a way that each portion of DMV output is not allowed to get value more than what has determined while dividing up the scale and also, these nonradial improvements are confined to fill the lower ranges before starting to fill upper ones which is in consistence with the logic of dividing up the scale in order to show the nonlinear behavior of DMV output.

Also, in the following model, let $w_{k_i+1} = 0$ and M be a positive constant "the big M". It is noteworthy to say that, s^{-k}_i is forced to zero by the binary variable w. The same is true in accordance with *Constrains (d)-(f)* for inputs which have nonlinear behavior. According to the predefined ranges for the i^th^ input where $i \in I_2$, nonradial improvements of inputs empty the ranges from the upper ones sequentially.

Since the defined tradeoff matrices for inputs have negative, zero and posi-

tive elements and $s^{-k}_i \geq 0$ it seems that s^{-k}_i may get its maximum value in a way that $x^k_{io} - s^{-k}_i \geq 0$. Thus, for overcoming this difficulty *Constraint (g)* is added to the model. It is noteworthy to say that there is no problem about reference set like what has been proved in about not identifying pareto efficient targets truly. Since nonradial improvements of inputs are confined to vary utmost up to the magnitude which fills the ranges. Obviously there is not such problem for outputs. Also

$$\delta_i = \text{Max}_i\{x_{ij} \mid \text{for all}_j\}, i \in I_1,$$
$$\delta^k_i = \text{Max}_i\{x^k_{ij} \mid \text{for all}_j\}, i \in I_2 \ , \ k = 1,...,k_i,$$
$$\gamma_r = \text{Max}_r\{y_{rj} \mid \text{for all}_j\}, r \in R_1,$$
$$\gamma^k_r = \text{Max}_r\{y^k_{rj} \mid \text{for all}_j\}, r \in R_2, \ k = 1,...,k_r.$$

As width of ranges are considered due to the opinion of experts, it is of importance to pay attention to this point that these intervals are better to be considered in a way that $\delta_i, , i \in I_1, \delta^k_i, i \in I_2, k = 1,...,k_i, \gamma_r, r \in R-1$ and $\gamma^k_r, r \in R_2, k = 1,...,k_r,$ have positive values. This is because of the structure of the objective function of the proposed model. It should be noted that in this case the objective function becomes unit invariant.

In such circumstances, the constant returns to scale form of the enveloping problem with f inputs, while both regular and IMV inputs are considered, and and g outputs, while both regular and DMV outputs are considered, is as follows:

$$
\min \frac{1-1/f(\sum_{i\in I_1}\frac{s_i^-}{\delta_i}+\sum_{i\in I_2}\sum_{k=1}^{K_i}\frac{s_i^{-k}}{\delta_i^k})}{1+1/g(\sum_{r\in R_1}\frac{S_r^+}{\gamma_r}+\sum_{r\in R_2}\sum_{k=1}^{K_r}\frac{S_r^{+k}}{\gamma_r^k})},
$$

$$
\begin{aligned}
&\sum_{j=1}^{n}\lambda_j x_{ij}\le x_{io}-s_i^-,\ \ i\in I_1,\\
&\sum_{j=1}^{n}\lambda_j x_{ij}^k+\sum_{l=1}^{f1}\tau_l P_{il}^k\le x_{io}^k-s_i^{-k},\ \ i\in I_2, k=1,\dots,k_{i,}\\
&\sum_{j=1}^{n}\lambda_j y_{rj}\ge y_{ro}+s_r^+,\ \ r\in R_1,\\
&\sum_{j=1}^{n}\lambda_j y_{rj}^k+\sum_{t=1}^{f2}\pi_t Q_{rt}^k\ge y_{ro}^k+s_r^{+k},\ \ r\in R_2\ ,\ k=1,\dots,k_r, && (9)\\
&s_r^{+k}\le y_{ro}^{-k}(1-v_{k-1}),\ r\in R_2,\ k=1,\dots,k_r, && (a)\\
&v_k\le(y_{ro}^{-k}-s_r^{+k}).M,\ r\in R_2,\ k=1,\dots,k_r, && (b)\\
&(y_{ro}^{-k}-s_r^{+k})\le v_k.M,\ r\in R_2,\ k=1,\dots,k_r, && (c)\\
&s_i^{-k}\le y_{ro}^{-k}(1-w_{k+1}),\ i\in I_2,\ k=1,\dots,k_i, && (d)\\
&w_k\le(x_{io}^{-k}-s_i^{-k}).M,\ i\in I_2,\ k=1,\dots,k_i, && (e)\\
&(x_{io}^k-s_i^{-k})\le w_k.M,\ i\in I_2,\ k=1,\dots,k_i, && (f)\\
&(x_{io}^k-s_i^{-k})\ge 0,\ i\in I_2,\ k=1,\dots,k_i, && (g)\\
&\lambda\ge 0\ ,\ \pi\ge 0,\ \tau\ge 0,\\
&s^-\ge 0,\ s^+\ge 0.
\end{aligned}
$$

From the condition that has been discussed since $\delta_i > 0$, $i\in I_1$, $\delta_i^k>0$, $i\in I_2, k=1,\dots,K_i\gamma_r>0, r\in R_1$ and $\gamma_r^k>0, r\in R_2, k=1,\dots,K_r$ and also $\lambda\ge 0$ it can be easily concluded that $x_i o\ge\delta_i^-, i\in I_1$ and $x_i o^k\ge\delta_i^{-k}, i\in I_2, k=1,\dots,K_i$. Therefore,

Theorem 1. The objective function for each feasible solution holds $0<\rho\le 0$.

Definition 1. A $DMU_o,(X_o,Y_o)$, is efficient if $\rho^*=1$.

The above mentioned condition is the same as having $s_i^{-*}=0, s_i^{-k*}=0, s_r^{+*}=0$, and $s_r^{+k*}=0$ for all i, r and k. Which means no excesses in regular and IMV inputs and no shortfall in regular and DMV outputs in any optimal solution.

Considering any optimal solution of the aforesaid model, the inefficient $DMU(X_O,Y_O)$ can be improved and become efficient(located onto the efficient frontier) be eliminating the input excess and increasing the output shortfall as follows:

$$x'_{io}=\sum_{j=1}^{n}\lambda_j^{*}x_{ij}+\delta_i^{*},\qquad i\in I_1,$$

$$x'^{k}_{io}=\sum_{j=1}^{n}\lambda_j^{*}x_{ij}^{k}+\sum_{l=1}^{f1}\tau_l^{*}P_{il}^{k}+\delta_i^{k*},\quad i\in I_2,\quad k=1,\dots,k_i,$$

$$y'_{ro}=\sum_{j=1}^{n}\lambda_j^{*}y_{rj}-\gamma_r^{*},\quad r\in R_1,$$

$$y'^{k}_{ro}=\sum_{j=1}^{n}\lambda_j^{*}y_{rj}^{k}+\sum_{t=1}^{f2}\pi_t^{*}Q_{rt}^{k}-\gamma_r^{k*},\quad r\in R_2\ ,\quad k=1,\dots,k_r.$$

Considering the above equalities the inefficient DMU_0 is projected onto the efficient frontier and this projection point is efficient. Consider the case in which with any optimal solution at hand the projection point is not efficient thus there exits a point in correspondence PPS which dominates it. Considering this point it is possible to have a feasible solution with the ob- jective function better than that of optimal solution which is a contradiction. therefore:

Theorem 2. The $DMU(x', y')$, is efficient.

From the above theorem it can be concluded that in any optimal solution the inequality constrains are binding.

4 | Example

Here, considering the presented nonradial model, an example with input-output data which are indicated in *Table 1* will be solved and the results are gathered in *Table 3*. The input and output vectors of six units are shown in *Table 1*. We consider three ranges for the second input and input as [0,20], [20,30] and [30,40] and for output as [0,30], [30,50] and [50,70], respectively. Also, corresponding variable for each portion of input and output are considered as v_{31},v_{32} and v_{33} and u_{31},u_{32} and u_{33}, respectively. In order to invoke the requirement that these variables should form a nonincreasing set of multipliers, for instance we impose on the multipliers u_{31}, u_{32}, u_{33} the constraints $4u_{32}\le u_{31}\le 8u_{32}$ and $4u_{33}\le u_{32}\le 8u_{33}$.

Moreover, considering in order to invoke the requirement that the corresponding variables should form a nondecreasing set of multipliers, for instance we impose on the multipliers v_{31}, v_{32}, v_{33} the constrains $1/4v_{22}\le v_{21}\le 1/2v_{22}$ and $1/4v_{23}\le v_{22}\le 1/2v_{23}$.

Table 1. Inputs.

DMUs	I1	I2	I3	I4	15	DMUs	I1	I2	I3	I4	I5
1	24	12	20	11	0	5	22	13	20	0	0
2	10	14	20	20	1	6	17	21	20	20	7
3	18	25	20	20	12	7	24	12	0	0	0

Table 2. Outputs.

DMUs	O1	O2	O3	O4	O5	DMUs	O1	O2	O3	O4	O5
1	60	46	30	20	14	5	53	65	30	20	11
2	55	62	30	20	5	6	65	45	30	10	0
3	50	58	30	17	0	7	60	46	30	20	14

Table 3. Results.

$\mathbf{DMU_s}$	θ^*	s_1^{-*}	s_2^{-*}	s_3^{-*}	s_4^{-*}	s_5^{-*}	s_1^{+*}	s_2^{+*}	s_3^{+*}	s_4^{+*}	s_5^{+*}
1	0.69	0	0	20	11	0	0	0	0	0	0
2	0.58	0	0	20	20	1	0	0	0	0	0
3	0.19	7.82	10.67	20	20	12	6	5.1	0	3	15
4	0.8	0	0	20	0	0	0	0	0	0	0
5	0.29	5.12	4.46	20	20	7	0	31.27	0	10	15
6	1	0	0	0	0	0	0	0	0	0	0

With the contribution of the bundles of *Constraints (a)-(f)* which are imposed into the model, nonradial improvements are confined in a way that each portion of IMV input and DMV output are expected to respectively lose and get values less than or equal to than what has determined while dividing up the scale and also, these nonradial improvements are confined to fill lower ranges before starting to fill upper ones for outputs and empty upper ranges before starting to empty lower ones. These two are in consistence with the logic of dividing up the scale in order to show the nonlinear behavior of IMV input and DMV output.

Now consider DMU_1, as it is shown in *Table 3* each portion has lost value according in their defined ranges from the upper interval. Also, here DMU_5 has lost value in a way that lower ranges are filled before starting to fill upper ones.

5 | Conclusion

In traditional DEA model, the aggregate output (input) has been deemed as a linear function of each output (input) but through real applications, linear pricing can not reveal the reality of situations in which there exists variables that have nonlinear impact on efficiency. In this paper, a nonradial piecewise linear CCR model has been considered in which nondecreasing and noninceasing set of multipliers for larger magnitude of inputs and outputs, respectively, describe the weight function. These variables are referred to as exhibiting increasing and DMVs. It is significant that these variables should form a nondecresing and nonincreasing sequences, thus special weight restrictions are imposed into the multiplier form. The incorporation of weight restrictions into the multiplier model, is the same as incorporating trade-offs into the envelopment model. In the current study, each divided portion of DMV output has been treated in a way that its related slack has not been allowed to get value more than what has determined while dividing up the scale into the mentioned ranges, and by imposing additional constrains which confine slack variables to get values sequentially in their specified ranges, a MIP model have been introduced in order to find the benchmark units and the reference set. considering the proposed model, it is possible to reveal the essence of nonlinearity of IMV input and DMV output on efficiency and truly find the pareto efficient targets. The significant feature of this model is that through solving one model the efficiency scores and also pareto efficient DMUs can be identified. The MIP model is based upon “ the big M” technique, therefore determining its magnitude in order achieve a feasible model is a significant decision which should be made with care. Further investigations of other concepts relevant to DEA can be considered from this point of view.

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Data Availability

The data that support the results of this study are derived from modeled scenarios and illustrative examples included within the manuscript. As such, no supplementary datasets have been generated or archived externally.

Conflicts of Interest

The authors have no financial, professional, or personal interests that could have influenced the outcomes or interpretations presented in this paper.

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