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Data Envelopment Analysis with Indices of Time Dependent

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Abstract

Traditional Data Envelopment Analysis (DEA) models measure the relative efficiency of a Decision-Maker Unit (DMU) with multiple inputs and outputs. One drawback of these models is the neglect of variable indices. In this paper, the indices are treated as time-varying variables, and we present methods for estimating efficiency and ranking DMUs.

Keywords: Data envelopment analysis, Decision maker unit.

1 | Introduction

The Data Envelopment Analysis (DEA) method is applied to evaluate the performance of Decision-Maker Units (DMUs). In this method, outputs are the services provided by an organization, and inputs are the facilities delivered to DMUs to enable better products and services. DMU efficiency is calculated by comparing the input and output indices across DMUs. One of the most critical DEA models is the CCR. Cooper and Rhodes first presented this model in [1].

This model is used for measuring the efficiency of an observed DMU, which is the organization to be evaluated by the ratio, a linear combination of outputs per linear combination of inputs, i.e., how well a DMU can convert its inputs into its outputs. Consider n DMUs each using m inputs, $Xj = (x1j, x2j, \ldots, xmj)$, to produce s output $Yj = (y1j, y2j, \ldots, ysj)$. Supposed weights of inputs and outputs are shown respectively in the form of $v\ (v1, \ldots, vm)$ and $u(u1, \ldots, us)$; thus, relative efficiency is defined as follows:

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$$RE_p = \max_{u,v} \frac{\dfrac{\sum_{r=1}^{s} u_r y_{rp}}{\sum_{i=1}^{m} v_i x_{ip}}}{\max_{1\le j\le n}\left\{\dfrac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}}\right\}}.$$

Nonlinear model of RE_P is transformed to a linear model as follows [1], [2]:

$$\begin{aligned}
&\text{CCR}: \max \ \sum_{r=1}^{m} u_r y_{rp},\\
&\text{s.t.}\\
&\sum_{i=1}^{m} v_i x_{ip} = 1,\\
&\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \ j = 1,\dots,n,\\
&u_j, v_i \ge \varepsilon, \quad i = 1,\dots,m, \ r = 1,\dots,s.
\end{aligned}$$

In some cases, indices are time-varying. In other words, they receive different amounts at different times. Because indices must be fixed in traditional DEA, models with variable data cannot be solved using traditional DEA methods, such as the CCR model [3–6]. The objective of this work is to propose a model and method for estimating efficiency and ranking such DMUs. Therefore, first, the best and worst times for each DMU are calculated, and then the efficiencies at the indices associated with the best and worst times are estimated. This paper is organized as follows. We introduce different methods in Section 2. Section 3 presents a method for estimating the relative efficiency and relative inefficiency of each DMU at the fixed time with respect to itself at other times. In Section 4, a method is delivered for finding the best and worst times. Section 5 introduces our methodology for solving the problem, based on indices for the best and worst times. In Section 6, a method for ranking DMUs is presented. A numerical example is given in Section 7, and Section 8 indicates the conclusion.

2 | The Decision Maker Units Efficiency with Variable Indices at Different Times

Suppose $y_{rP}(t)$ is a given level of rth output of *p*th DMU at the time of t, likewise, $x_{iP}(t)$ is a given level of *i*th input of P.th DMU at the time, t. There are m inputs, s outputs, n DMUs, and the domain of t is the interval [a, b]. We want to estimate the relative efficiency of DMUp with respect to other DMUS. The calculating methods can be estimated for efficiency at different times. We define efficiency at time t.

Definition 1. It is said that DMUp is efficient at time t if the optimal solution to the model below is 1. And it is efficient when maxθ(t) equals 1.

$$\begin{aligned}
&\text{CCR}(t): \theta(t) = \max \ \sum_{r=1}^{m} u_r y_{rp}(t),\\
&\text{s.t.}\\
&\sum_{i=1}^{m} v_i x_{ip}(t) = 1,\\
&\sum_{r=1}^{s} u_r y_{rj}(t) - \sum_{i=1}^{m} v_i x_{ij}(t) \le 0, \qquad j = 1,\dots,n,\\
&u_j, v_i \ge \varepsilon, \qquad i = 1,\dots,m, \ r = 1,\dots,s.
\end{aligned}$$

The problems with the definition mentioned above are summarized as follows: Each DMU has the best efficiency at a time that may differ from that of the other DMUs. So, comparing DMUs at a fixed time is neither just nor logical. For example, suppose the indices of three DMU are as follows.

Table 1. Efficiency scores over time.

DMU	Input s			Output s			Best Days for DMUs
	First Day	Second Day	Third Day	First Day	Second Day	Third Day	
1	15	7	11	16	22	11	First day
2	22	4	14	9	41	11	Second day
3	31	11	7	11	17	51	Third day

On the first day, the efficient DMU is DMU1; on the second and third days, the results are DMU2 and DMU3, respectively. Therefore, according to the above method, each of the three DMUs is efficient. Still, under the condition that the indices associated with DMU 1 are considered the first and second days for DMU 2, and the third day for DMU 3, it results in DMU 1 being inefficient and DMU 2 and 3 being efficient.

Considering the summation of indices can be another method for estimating the efficient DMU.

Definition 2. It is said that DMUp is fully efficient if the optimal solution to the following model is 1.

$$
\begin{aligned}
&\text{ACCR}: \max \ \sum_{r=1}^{m} u_r \int_a^b y_{rp}(t)dt,\\
&\text{s.t.}\\
&\sum_{i=1}^{m} v_i \int_a^b x_{ip}(t)dt = 1,\\
&\sum_{r=1}^{s} u_r \int_a^b y_{rj}(t)dt - \sum_{i=1}^{m} v_i \int_a^b x_{ij}(t)dt \le 0, \quad j = 1,...,n,\\
&u_j, v_i \ge \varepsilon, \quad i = 1,...,m, \ \ r = 1,...,s.
\end{aligned}
$$

Total efficiency can't be convenient, because it is not considered a relation between indices; moreover, one DMU may be total efficient, but it is not efficient at any time. To clarify the subject, consider the following example: Consider three DMUs with one input and one output, as shown in the table below:

Table 2. Three DMUs with one input and one output.

	First Day		Second Day		Third Day		Forth Day		Sum of the Outputs	
DMU	Input	Output	Input	Output	Input	Output	Input	Output	Input	Output
1	1	10	2	6	10	3	2	5	15	24
2	5	7	4	5	3	7	3	6	15	25
3	7	8	4	2	2	8	2	6	15	24

It is seen that on the first and second days, DMU1 is efficient; on the third and fourth days, DMU2 is efficient; and throughout the period, DMU3 is efficient, although it never reached the efficient frontier.

It seems that, to estimate DMUs' efficiency, it is convenient that the DMUs be compared at the best times and likewise at the worst times. Attention: the best time for each DMU differs from that of other DMUs. Likewise, the worst times for DMUs differ. Now we propose a method based on the best and worst times for each DMU. The method is to compare the indices associated with the best time of each DMU with those of other DMUs. In the same way, indices associated with the worst time of each DMU are compared with indices associated with the best time of other DMUs. This method is summarized into four steps.

Step 1. It is the relative efficiency of a DMU at a given time with respect to other times for the same DMU.
Step 2. Is finding the best and worst times for all of the DMUs.

Step 3. Is finding the efficiency of each DMU based on the best and the worst times.

Step 4. Is ranking DMUs. In the parts below, each one of the above-mentioned steps is explained.

3 | Relative Efficiency and Inefficiency of a Decision-Maker Unit in Fixed Time vs. Other Times

Let yrP(t) and xiP(t) be respectively the given levels of rth output and ith input of DMUP at time t. In other words, indices of DMUP are variable in the interval of [a, b]. There are m inputs, s outputs, and the domain of t is the interval of [a, b]. We suppose that yrP(t) and xiP(t) are continuous and nonnegative functions. In this part, we suggest a method for estimating the relative efficiency of a DMU at a fixed time vs. the same DMU at other times. It means any time is considered a DMU. We define DT(P,t0) as a DMU with inputs $x_{ip}(t_0),\ i=1,\ldots,m$ and outputs $y_{rp}(t_0),\ r=1,\ldots,s$. Likewise, we define $DT(P)=\{DT(P,t) \mid t\in[a,b]\}$. Thus the relative efficiency DT (P,t0) vs. Deterministic Time D(T)(P) is defined as follows:

$$RE_p(t_0)=\theta=\max\frac{\dfrac{\sum_{r=1}^{s}u_r y_{rp}(t_0)}{\sum_{i=1}^{m}v_i x_{ip}(t_0)}}{\max\limits_{t\in[a,b]}\left\{\dfrac{\sum_{r=1}^{s}u_r y_{rp}(t)}{\sum_{i=1}^{m}v_i x_{ip}(t)}\right\}},\quad u,v\geq\varepsilon.$$

Because of

$$\left\{\frac{\sum_{r=1}^{s}u_r y_{rp}(t)}{\sum_{i=1}^{m}v_i x_{ip}(t)}\right\}.$$

If it continues on the interval [a, b], we can write:

$$\max_{t\in[a,b]}\left\{\frac{\sum_{r=1}^{s}u_r y_{rp}(t)}{\sum_{i=1}^{m}v_i x_{ip}(t)}\right\}=1/w\ .$$

Thus, we have:

$$RE_p(t_0)=\max_{u,v}\frac{\sum_{r=1}^{s}u_r w y_{rp}(t_0)}{\sum_{i=1}^{m}v_i x_{ip}(t_0)},$$

s. t.

$$\frac{\sum_{r=1}^{s}u_r y_{rp}(t)}{\sum_{i=1}^{m}v_i x_{ip}(t)}\leq\frac{1}{w},\ t\in[a,b].$$

$$u_r,v_i\geq\varepsilon,\ r=1...s,\ i=1,...,m.$$

By substituting $Y_{rp}(t_0)$ instead of $wy_{rp}(t_0)$, the above model is transformed as follows:

$$\text{FCCRT}: \max_{u,v} \frac{\sum_{r=1}^{s} u_r Y_{rp}(t_0)}{\sum_{i=1}^{m} v_i x_{ip}(t_0)},$$

s. t.

$$\sum_{r=1}^{s} u_r Y_{rp}(t) - \sum_{i=1}^{m} v_i x_{ip}(t) \le 0, \qquad t \in [a,b],$$
$$u_r, v_i \ge \varepsilon,\ \ r = 1 \ldots s,\ \ i = 1,\ldots,m.$$

Thus, by putting $\sum_{i=1}^{m} v_i x_{ip}(t_0) = \frac{1}{q}$ and $X_{ip}(t_0) = q x_{ip}(t_0)$ in FCCRT, the following model will result:

$$\text{CCRT}: RE_p(t_0) = \theta = \max_{u} \sum_{r=1}^{s} u_r Y_{rp}(t_0),$$

s. t.

$$\sum_{i=1}^{m} v_i X_{ip}(t_0) = 1,$$
$$\sum_{r=1}^{s} u_r Y_{rp}(t) - \sum_{i=1}^{m} v_i x_{ip}(t) \le 0, \qquad t \in [a,b],$$
$$u_r, v_i \ge \varepsilon,\ \ r = 1 \ldots s,\ \ i = 1,\ldots,m.$$

The Above model is the same CCR model, with an infinite number of DMUs. We profess that the optimum solution of the model below is an upper bound of the optimum solution of Categorical and Combinatorial Representation Theory (CCRT):

$$\text{BCCRT}: \theta = \max_{u} \sum_{r=1}^{s} u_r Y_{rp}(t_0),$$

s. t.

$$\sum_{i=1}^{m} v_i X_{ip}(t_0) = 1,$$
$$\sum_{r=1}^{s} u_r \int_a^b Y_{rp}(t)dt - \sum_{i=1}^{m} v_i \int_a^b x_{ip}(t)dt \le 0,$$
$$u_r, v_i \ge \varepsilon,\ \ r = 1 \ldots s,\ i = 1,\ldots,m.$$

Theorem 1. Suppose that $(u*, v*, \theta_p)$ and $(\bar{u}^*, \bar{v}^*, \bar{\theta}_p)$ are respectively optimum solutions of CCRT, BCCRT models. Therefore

$$\theta_p(u^*, v^*) \le \bar{\theta}_p(\bar{u}^*, \bar{v}^*).$$

Proof: The logic below shows that (u*, v*) is a feasible solution to BCCRT.

$$\sum_{r=1}^{s} u_r^*\ Y_{rp}(t) - \sum_{i=1}^{m} v_i^* x_{ip}(t) \le 0\ \ t \in [a,b]\ \Rightarrow \int_a^b \left(\sum_{r=1}^{s} u_r^* Y_{rp}(t) - \sum_{i=1}^{m} v_i^* x_{ip}(t) \right) dt \le 0,$$
$$\Rightarrow \sum_{r=1}^{s} u_r^* \int_a^b Y_{rp}(t)dt - \sum_{i=1}^{m} v_i^* \int_a^b x_{ip}(t)dt \le 0.$$

Therefore, we will have:

$$\theta_p = \sum_{r=1}^{s} u_r^* Y_{rp}(t_0) \le \sum_{r=1}^{s} \bar{u}_r^* Y_{rp}(t_0) = \bar{\theta}_p.$$

Theorem 2. Suppose $(\bar{u}^*, \bar{v}^*, \bar{\theta}_p)$ it is the optimum solution of the BCCRT model and

$$\max_{a \le t \le b}\left(\sum_{r=1}^{s} \bar{u}_r^* y_r(t) - \sum_{i=1}^{m} \bar{v}_i^* x_i(t)\right) = f(t^*) \le 0.$$

Therefore, it is an optimal solution of the CCRT model.

Proof: Suppose θ_p is a value of optimum associated with the objective function of the CCRT model, then according to *Theorem 1* $\theta_p \le \bar{\theta}_p$. Also, because $f(t*)$ is less than or equal to zero, for each t we have:

$$\sum_{r=1}^{s} \bar{u}_r^* y_r(t) - \sum_{i=1}^{m} \bar{v}_i^* x_i(t) \le 0.$$

Thus $(\bar{u}^*, \bar{v}^*, \bar{\theta}_p)$, it is a feasible solution of the CCRT model, and as a result, we have $\bar{\theta}_p \le \theta_p$. Therefore, it is clear that $\bar{\theta}_p = \theta_p$ it $(\bar{u}^*, \bar{v}^*, \bar{\theta}_p)$ is an optimum solution of the CCRT model. Now we present an Algorithm for solving the CCRT model by using the linear model of BCCRT.

Algorithm of categorical and combinatorial representation theory

Step1. Solve the linear model of BCCRT and obtain $(\bar{u}^*, \bar{v}^*, \bar{\theta}_p)$ an optimum solution of the BCCRT model and an upper bound for the solution space of the CCRT model.

Step 2. Obtain the maximum value of the function that is defined as follows:

$$F(t) = \left(\sum_{r=1}^{s} \bar{u}_r^* y_r(t) - \sum_{i=1}^{m} \bar{v}_i^* x_i(t)\right).$$

Suppose the maximum value of the above function occurs at t^*, if

$$\sum_{r=1}^{s} \bar{u}_r^* y_r(t^*) - \sum_{i=1}^{m} \bar{v}_i^* x_i(t^*) \le 0.$$

Then, it is an optimal solution of the CCRT model. Otherwise, we add the following constraint to the BCCRT model.

$$\sum_{r=1}^{s} u_r y_r(t^*) - \sum_{i=1}^{m} v_i x_i(t^*) \le 0.$$

Then return to *Step 1*. For estimating the relative inefficiency in $DT(P, t0)$. We apply a similar method above based on the definition of inefficiency as follows:

$$IRE_p(t_0)=1/\min_{u,v}\frac{\dfrac{\sum_{r=1}^{s}u_r y_{rp}(t_0)}{\sum_{i=1}^{m}v_i x_{ip}(t_0)}}{\min\limits_{t\in[a,b]}\left\{\dfrac{\sum_{r=1}^{s}u_r y_{rp}(t)}{\sum_{i=1}^{m}v_i x_{ip}(t)}\right\}},\quad u,v\geq\varepsilon.$$

Similar to what was said, models of CCRT and BCCRT are transformed to the Integrated Common Complex Root Test (ICCRT)and IBCCRT as follows.

$$\begin{aligned}
&ICCRT:(1/IRE_p(t_0))=1/\theta=\min_u\sum_{r=1}^{s}u_r Y_{rp}(t_0),\\
&\text{s. t.}\\
&\sum_{i=1}^{m}v_i X_{ip}(t_0)=1,\\
&\sum_{r=1}^{s}u_r Y_{rp}(t)-\sum_{i=1}^{m}v_i x_{ip}(t)\geq 0\quad t\in[a,b],\\
&u_r,v_i\geq\varepsilon,\quad r=1...s,\ i=1,...,m.
\end{aligned}$$

$$\begin{aligned}
&IBCCRT:\min_u\sum_{r=1}^{s}u_r Y_{rp}(t_0),\\
&\text{s. t.}\\
&\sum_{i=1}^{m}v_i X_{ip}(t_0)=1,\\
&\sum_{r=1}^{s}u_r\int_a^b Y_{rp}(t)dt-\sum_{i=1}^{m}v_i\int_a^b x_{ip}(t)dt\geq 0,\\
&u_r,v_i\geq\varepsilon,\quad r=1...s,\ i=1,...,m.
\end{aligned}$$

Then, the algorithm CCRT is transformed into the ICCRT algorithm**.**

Algorithm of the integrated common complex root test

Step 3. Solve the linear model of IBCCRT and obtain $(\bar{u}^*,\bar{v}^*,\bar{\theta}_p)$ an optimum solution of the IBCCRT model and a lower bound for the solution space of the ICCRT model.

Step 4. Obtain the minimum value of the function that is defined as follows:

$$F(t)=(\sum_{r=1}^{s}\bar{u}_r^* y_r(t)-\sum_{i=1}^{m}\bar{v}_i^* x_i(t)).$$

Suppose the minimum value of the above function happens at t^*, if

$$\sum_{r=1}^{s}\bar{u}_r^* y_r(t^*)-\sum_{i=1}^{m}\bar{v}_i^* x_i(t^*)\geq 0.$$

Then $(\bar{u}^*,\bar{v}^*,\bar{\theta}_p)$, it is the optimum solution of the ICCRT model. Otherwise, we add the following constraint to the IBCCRT model,

$$\sum_{r=1}^{s} u_r y_r(t^*) - \sum_{i=1}^{m} v_i x_i(t^*) \ge 0.$$

Then return to *Step 3*.

4 | The Most Efficient and Inefficient Time for a Decision-Maker Unit

In this part, we present a method for obtaining the most efficient and the least efficient DT(P). Suppose we want to estimate the time at which it has maximum efficiency in a bank. One of the input indices can be employees' scores; likewise, investment at short and long times can be listed as output indices. The above indices are changed each time, whereas we have only the values of the indices for a finite days. Therefore, for each index that is variable, we fit a piecewise linear function to the data. In other words, we make a function by drawing a line between each pair of adjacent points. However, we make a continuous function for each index; it is variable, but actually, the variation of indices, in the length of day, is almost very little. Therefore, we can estimate the relative efficiency of a DMU in a time interval of a day. In other words, we divide the time interval [a, b] into small parts with length Δ, and we estimate relative efficiency at partition points. Now we present an algorithm for obtaining the best and the worst member of DT (P).

Algorithm best (worst) member of deterministic time (P)

Step 5. Fit a piecewise linear function for each variable index.

Step 6. Let

$$t_i = a + i\Delta \ , \ i = 0,\dots,\frac{b\text{-}a}{\Delta},$$

Is partition of [a b], for any t_i , we apply the Algorithm of ICCRT, for estimating the relative efficiency (inefficiency) of DT(P, t_i).

Step 7. For any efficient (inefficient) member of DT (P) obtained at *Step 6*, a ranking method is provided.

4.1 | A Ranking Method for Efficient Members of Deterministic Time (P)

If the number of efficient members of DT (P) is more than one, we must apply one of the ranking methods [7–11]. In here, we propose a new ranking method. At first, we define a Virtual DMU, for any DMUj, as T-DMUj, and it is defined as follows:

$$\text{inputs of T-DMU}_j : \int_a^b x_{ij}(t)dt, \quad i = 1,\dots,m,$$

$$\text{outputs of T-DMU}_j : \int_a^b Y_{rj}(t)dt, \quad r = 1,\dots,s.$$

Suppose a DT (P,t_0) is efficient at *Step 5* and *Step 6* of the algorithm's best member of DT (P). Therefore, t_0 is an efficient time for DMUp. But the best time is when it is possible to achieve high efficiency in competition with other DMUs. Therefore, we calculate the efficiency of the DT (P, t_0) $T - DMU_j, \ j = 1,\dots,n \ j \neq p$ based on the CCR model. It is named the RCCRT model.

$$\begin{aligned}
&\text{RCCRT}: S(DT(P,t_0)) = \max_u \sum_{r=1}^{s} u_r Y_{rp}(t_0),\\
&\text{s. t.}\\
&\sum_{i=1}^{m} v_i X_{ip}(t_0) = 1,\\
&\sum_{r=1}^{s} u_r \int_a^b Y_{rj}(u)du - \sum_{i=1}^{m} v_i \int_a^b x_{ij}(u)du \le 0, \quad j = 1,...,n,\ j \ne p,\\
&u_r, v_i \ge \varepsilon,\ \ r = 1...s,\ i = 1,...,m.
\end{aligned}$$

For any efficient DT (P,t), the RCCRT model is applied. Each one that maximizes the objective function is the best, and this is shown as best-DT (P). S (Best-DT (P)) is not equal to one because it is not on the efficiency frontier. Also, if DT (P,t_0) is Best-DT (P), then t_0 will be named Best-time(p).

4.2 | Ranking Inefficient Deterministic Time

If the number of inefficient members of DT (P) is more than one, we similarly apply the ranking method as follows:

$$\begin{aligned}
&\text{IRCCRT}: IS(DT(P,t_0)) = \min_u \sum_{r=1}^{s} u_r Y_{rp}(t_0),\\
&\text{s. t.}\\
&\sum_{i=1}^{m} v_i X_{ip}(t_0) = 1,\\
&\sum_{r=1}^{s} u_r \int_a^b Y_{rj}(t)dt - \sum_{i=1}^{m} v_i \int_a^b x_{ij}(t)dt \ge 0, \quad j = 1,...,n,\ j \ne p.\\
&u_r, v_i \ge \varepsilon,\ \ r = 1...s,\ i = 1,...,m.
\end{aligned}$$

For any inefficient DT (P,t), the IRCCRT model is applied. Each one gaining the least objective function is the worst and is shown as worst-DT (P). It is obvious that IS (worst -DT (P)) is not equal to one because it is not on the efficiency frontier. Also, if DT (P,t_0) is worst-DT (P), then t_0 will be named worst-time(p).

5 | Efficiency of Decision-Maker Units with Functional Index

Now we want to calculate the efficiency of DMUp vs. all of the DMUs. We use the CCR model once when indices are associated with best-DT (k) and once separately at worst-DT (k) for each DMU_k. The efficiency of the DMUs will be specified by the following definitions.

Definition 3. Suppose that, for each DMUk, the indices have been considered as best-DT(k). Also, suppose DMU_p is the unit under evaluation. Therefore, the optimal value of the objective function of the CCR model is called the relative efficiency at best times and is denoted by θ_P(best).

Definition 4. Suppose for each DMU_k, indices are associated with worst-DT (k). Also, suppose DMU_p is the unit under evaluation. Therefore, the optimal value of the objective function of the CCR model is called the relative efficiency at worst times and is denoted by θ_P(worst). Thus, by solving CCR models according to the above-mentioned definitions, we obtain θ_P(best) and θ_P(worst) for each DMU. Now we define an efficiency range for DMU_P.

Definition 5. The range of efficiency of *the P*th unit is defined as follows:

$$\begin{aligned}
&\mathrm{RP} = \min\big(\theta P(\mathrm{best}), \theta P(\mathrm{worst})\big),\\
&\max\big(\theta P(\mathrm{best}), \theta P(\mathrm{worst})\big).
\end{aligned}$$

Definition 6. If R_p is equal to [1,1], then DMU_p is efficient (E). But if R_p is equal to [a<1,1] then DMU_p is a Weakly Efficient unit (WE) and otherwise it is inefficient (IE).

6 | Ranking

For ranking the DMUs, efficient units (E), WE units, and inefficient units (IE) are placed respectively. Ranking efficient DMUs (E) will be done by the following models.

$$
\begin{aligned}
&\text{RSCCR}: \text{Rs(p)} = \max_{u} \sum_{r=1}^{s} u_r yw_{rp},\\
&\text{s. t.}\\
&\sum_{i=1}^{m} v_i xw_{ip} = 1,\\
&\sum_{r=1}^{s} u_r yb_{rj} - \sum_{i=1}^{m} v_i xb_{ij} \le 0,\\
&u_r, v_i \ge \varepsilon, \quad r = 1 \ldots s, \quad i = 1, \ldots, m.
\end{aligned}
$$

In which yw_{rp} xw_{ip} are respectively output and input indices associated with worst-DT (p), and yb_{rj} xb_{ij} are respectively output and input indices associated with best-DT (k). Rs (P) is not equal to one because the indices of DMUp are not on the efficiency frontier. Note that if Rs(P) is greater than Rs(q), then DMUp is more efficient than DMUq. Ranking WE units will be done according to min (θ_P(best), θ_P(worst)), and finally, the ranking of inefficient units was ordered according to the mean of RP.

$$
\text{mean}(R_p) = \frac{\min(\theta_p(\text{best}), \theta_p(\text{worst})) + \max(\theta_p(\text{best}), \theta_p(\text{worst}))}{2}.
$$

7 | Numerical Example

We perform the above method at twenty banks. We consider four inputs (area, archaism, employee score, and equipment) and five outputs (Loan savings, other deposits, loan current, long-term investment, and short-term investments). Area, archaism, and equipment are constant, but other indices are variable. Amounts of indices have been specified in thirty-eight months, and at the end of each month. For each unit, a fixed piecewise linear variable index is used. Thus, we have six curves for each unit and a total of one hundred twenty curves. We select the best (worst) member of DT (P) by setting Δ=1. It means the criterion of time is one day. *Table 1* shows data associated with DMU one. The constant indices of DMU_1 are eight, three thousand, five hundred eighty one million and five hundred fifty two thousand three hundred thirty seven point of three respectively for archaism, area, and equipment.

The piecewise linear functions of indices of DMU1 have been shown in *Fig. 1*. Functional indices of employees' score, associated with twenty DMUs, have been shown in *Fig. 2*. We apply the algorithm to find the best (worst) member of DT (P) for any DMU. *Table 2* shows the best times and the worst times of DMUs in thirty-eight months or one thousand one hundred fifty-seven days. Now, indices are obtained at the best and worst times using piecewise linear functions. Then we solve the CCR model using best- and worst-time DMU scores, and thereafter rank the DMUs and solve the RSCCR model (See *Table 3*).

Table 1. Indices of DMU 1 for 38 months.

Month	Employee Score	Loan Savings	Other Deposits	Loan Current	Long-Term Investment	Short-Term Investments
1	340.99741	15156441131	5.93478E+11	5.023E+11	9.81138E+11	1.29816E+11
2	340.6133054	15486150847	4.79739E+11	5.749E+11	9.75238E+11	1.29178E+11
3	336.8751886	15147750063	5.99063E+11	4.912E+11	9.72861E+11	1.32733E+11
4	341.6745722	16227802935	6.34443E+11	5.664E+11	9.67165E+11	1.25776E+11
5	341.7932867	16329743466	6.69817E+11	5.021E+11	6.61973E+11	1.32697E+11
6	338.2901693	18426738249	6.52629E+11	4.936E+11	6.71007E+11	1.35828E+11
7	340.7772709	18760144321	6.04068E+11	6.56E+11	6.91833E+11	1.36808E+11
8	338.7538205	18551494956	5.89279E+11	6.756E+11	7.01078E+11	1.42731E+11
9	336.8991148	18845106575	5.03993E+11	6.19E+11	7.04233E+11	1.59779E+11
10	335.4943076	19516960731	3.73349E+11	5.136E+11	7.03186E+11	1.56556E+11
11	350.9977094	20583912521	4.36225E+11	5.472E+11	7.08022E+11	1.69354E+11
12	348.8346741	18897219011	1.55483E+12	4.868E+11	7.2697E+11	1.56846E+11
13	347.5338595	18572006313	5.09177E+11	5.061E+11	7.34424E+11	1.6576E+11
14	345.3115685	19825267128	4.1415E+11	4.647E+11	7.43651E+11	1.81348E+11
15	340.0823682	19592206863	6.4238E+11	5.793E+11	7.55343E+11	1.90706E+11
16	335.9379133	19730960702	5.52352E+11	4.05E+11	7.72313E+11	1.90767E+11
17	339.1593231	22229233369	3.11657E+11	3.787E+11	7.64483E+11	2.11048E+11
18	339.7501537	21208703053	9.52745E+11	4.718E+11	9.25755E+11	2.38202E+11
19	365.9365962	21992321718	3.89493E+11	4.777E+11	7.90713E+11	1.9374E+11
20	370.5325774	21411480765	3.67521E+11	4.687E+11	7.99221E+11	1.99317E+11
21	372.1248687	25164820317	6.89624E+11	9.547E+11	8.62069E+11	2.21567E+11
22	375.2883735	24571771182	5.78908E+11	4.37E+11	8.69276E+11	2.01552E+11
23	372.5527615	22894085689	2.86754E+11	3.914E+11	8.64886E+11	2.10385E+11
24	371.4020233	25241207595	8.31076E+11	4.754E+11	8.55755E+11	2.18468E+11
25	367.9021869	24890509495	9.43842E+11	4.754E+11	8.57446E+11	2.26739E+11
26	366.8483134	24785206500	4.82745E+11	4.483E+11	8.94939E+11	2.34361E+11
27	365.2892351	22930702673	4.10299E+11	4.323E+11	9.6918E+11	2.4576E+11
28	365.9896226	23735097146	3.9839E+11	4.885E+11	9.18187E+11	2.39287E+11
29	374.5125005	21795966355	6.32914E+11	3.682E+11	9.23015E+11	2.30984E+11
30	379.1146945	21208703053	9.52745E+11	9.23E+11	9.25755E+11	2.38202E+11
31	374.1857602	21083602715	5.83187E+11	4.802E+11	9.42186E+11	2.3064E+11
32	376.4482101	20726257952	6.19136E+11	1.257E+12	1.1484E+12	2.37724E+11
33	369.9861592	21129797402	6.1052E+11	8.581E+11	1.13333E+12	3.33788E+11
34	366.5546557	21763727220	1.15892E+12	8.525E+11	1.15639E+12	3.18003E+11
35	364.3849617	26343660128	9.29498E+11	1.567E+12	1.17009E+12	3.17454E+11
36	367.9442361	25046032311	1.83154E+12	1.516E+12	1.20084E+12	3.1153E+11
37	360.0290417	27116587637	9.15333E+11	3.332E+12	1.36597E+12	3.50474E+11
38	359.0287691	27717975406	9.74372E+11	1.513E+12	1.44586E+12	3.60855E+11

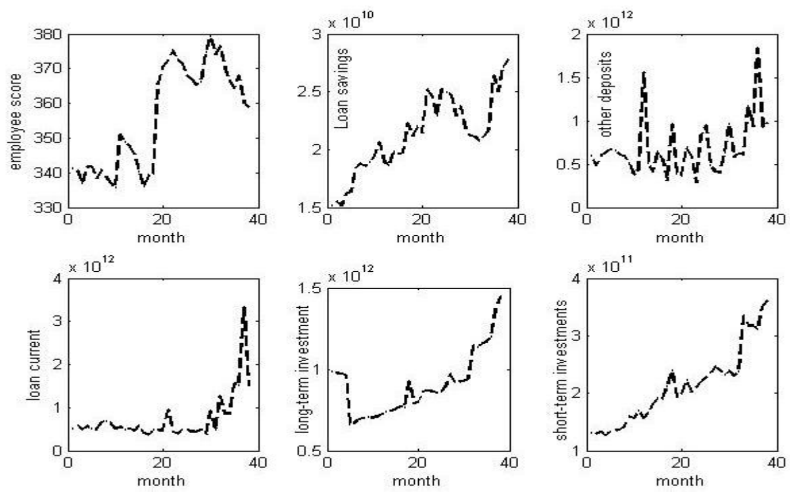

Fig. 1. Curves associated with functional indices of DMU 1.

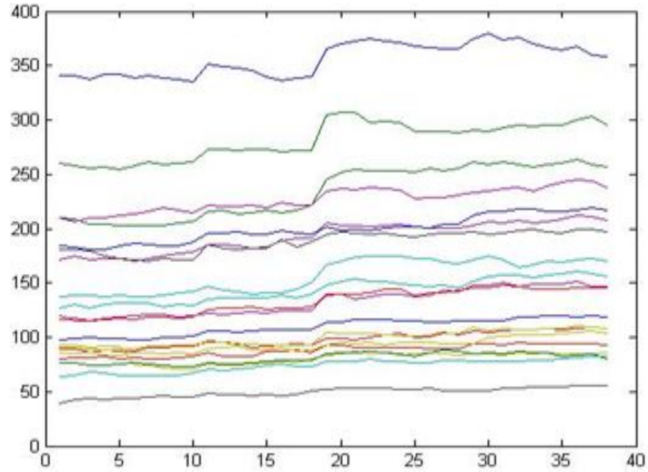

Fig. 2. Functional indices of employees' scores associated with twenty DMUs.

Table 2. Best times and worst times for 12 banks at 1…1157 days.

DMUs	Worst Times	Best Times
DMU1	883	58
DMU2	944	153
DMU3	974	436
DMU4	700	303
DMU5	548	181
DMU6	670	279
DMU7	487	461
DMU8	91	170
DMU9	1126	1083
DMU10	122	126
DMU11	792	44
DMU12	731	276
DMU13	61	104
DMU14	487	24
DMU15	731	705
DMU16	640	16
DMU17	883	66
DMU18	1127	551
DMU19	1096	244
DMU20	853	611

Table 1. Efficiency of DMUs and ranking.

DMUs	Efficiency at Best- Time	Efficiency at the Worst Time	Mean	Rs(P)	Efficiency Position	Final Rank
1	1	1	1	1.4589	E	6
2	1	1	1	0.8348	E	8
3	1	1	1	1.7959	E	5
4	1	1	1	0.5003	E	11
5	1	1	1	32	E	1
6	1	0.9369	0.96845	---------	WE	13
7	1	1	1	3.9248	E	3
8	1	1	1	2.2644	E	4
9	1	1	1	1.1112	E	7
10	1	1	1	7.7514	E	2
11	1	0.2563	0.62815	---------	WE	20
12	1	0.4398	0.7199	---------	WE	16
13	1	0.4240	0.712	---------	WE	17
14	1	0.3834	0.6917	---------	WE	18
15	1	1	1	0.62	E	10
16	1	1	1	0.6425	E	9
17	1	1	1	0.2931	E	12
18	1	0.8432	0.9216	---------	WE	14
19	0.5201	0.3605	0.4403	---------	IE	19
20	0.4671	0.6865	0.5768	---------	IE	15

8 | Conclusion

In this paper, we analyzed a new DEA in which some indices are time-dependent.

For the calculation of the relative efficiency of the units, we selected indices in the following forms: 1) Indices associated with the best time of each unit are selected. The best time of a unit is the time at which the unit is in its best position. On the other hand, if we suppose indices associated with different times of a unit make the virtual units, then the virtual unit associated with the best time is efficient against all of the virtual units. In fact, we can say that in this form, indices are selected by the own DMU. Eventually, the efficiency of units is calculated at best times. 2) Indices associated with the worst time of each unit are selected. The worst time for a unit is the time that it is in the worst position. On the other hand, the virtual unit associated with the worst time is inefficient against all of the virtual units. In fact, in this form, the indices of a DMU are selected by the other DMUs. Thus, the efficiency of units is calculated at worst times, and 3) Indices associated with the unit under evaluation are selected at best times, and indices associated with other units are selected at worst times. This form is applied for ranking.

Finally, for each unit, we have a range of efficiency that has been obtained at Form 1 and Form 2. Therefore, units are divided into efficient units, WE units, and inefficient units. Efficient units are efficient at both forms. WE units are only efficient in one form. Inefficient units are inefficient in both forms. For ranking, efficient units are ranked first according to Form 3. The next time, WE units are ranked according to the minimum efficiency associated with form 1 or form 2. At the end of time, inefficient units are ranked according to the mean of efficiency associated with form 1 or form 2.

Conflict of Interest

The authors declare that they have no conflict of interest regarding the publication of this manuscript.

Data Availability

The datasets used and/or analyzed during the current study are available from the corresponding author on reasonable request.

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