Research Annals of Industrial and Systems Engineering

www.raise.reapress.com

Res. Ann. Ind. Syst. Eng. Vol. 3, No. 3 (2026) 182–191.

Paper Type: Original Article

Fractional Order Efficiency: A New DEA Paradigm Based on Memory-Dependent Dynamics

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Citation:

Received: 13 March 2026 *Revised: 15 May 2026* *Accepted: 01 July 2026*	Abdolmaleki, E., Edalatpanah, S. A., Yahyapour, M. T., & Joorbonyan, Z. (2026). Fractional order efficiency: A new DEA paradigm based on memory-dependent dynamics. *Research Annals of Industrial and Systems Engineering*, *3*(3), 182-191.

Abstract

Classical Data Envelopment Analysis (DEA) is largely static: It benchmarks Decision Making Units (DMUs) using momentary observations. However, many manufacturing systems, especially energy systems, exhibit memory and path dependence, where current performance depends on historical paths. In parallel, uncertainty in energy data is often epistemic and multi-source, which is naturally modeled by fuzzy sets. This paper introduces fractional-order efficiency by embedding DEA in a dynamic framework driven by Fuzzy Fractional Differential Equations (FFDE) in the sense of Caputo. We define efficiency as a functional input-output path with a fractional memory kernel, derive best (worst) case efficiency bounds under fuzzy parameter uncertainty, and develop a convergence theorem that connects the proposed fractional-order efficiency to a subset of discrete window DEA scores. A repeatable numerical study inspired by a regional power source demonstrates: How fractional-order controls inefficiency inertia (memory), how fuzzy uncertainty yields efficiency intervals, and how the method produces sustainability rankings under noisy multi-year data.

Keywords: Fractional calculus, Data envelopment analysis, Fuzzy data envelopment analysis, Memory effects, Energy efficiency.

1 | Introduction

Data Envelopment Analysis (DEA) has become one of the most influential nonparametric methods for measuring the relative efficiency of Decision-Making Units (DMUs) in economics, operations research, and energy systems [1], [2]. Since the CCR and BCC models [2], DEA has evolved into a comprehensive analytical

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https://doi.org/10.22105/raise.v3i3.103

framework including slack-based measures, network DEA, dynamic DEA, and window analysis, which allow for a more realistic representation of production and service systems [1], [2]. DEA-based approaches have been widely applied in energy efficiency, environmental performance, and sustainability assessment, demonstrating their methodological flexibility and strong theoretical foundations [3–5]. However, despite these advances, most DEA models remain essentially static or quasi-dynamic, relying on cross-sectional images or rolling windows that implicitly assume weak temporal dependence between successive observations [6], [7]. In practical energy and sustainability systems, this assumption is rarely valid. Power generation units, renewable energy infrastructures, and industrial production systems have strong path dependence resulting from capital rigidity, learning-by-doing effects, technology lock-in, regulatory inertia, and cumulative investment decisions [6], [8]. As a result, current efficiency levels are strongly influenced by historical input-output paths rather than by instantaneous realizations [8], [9].

Although dynamic DEA and window DEA partially account for intertemporal structures, they are usually formulated in integer-order Markovian frameworks that ignore long-term memory and heritability effects [6], [9]. In parallel, efficiency assessment in energy systems is inherently affected by epistemic uncertainty. Energy input and output data are often imprecise due to measurement errors, aggregation procedures, reporting delays, and reliance on expert judgment-based estimates [10]. To address this issue, fuzzy set theory has been widely integrated into DEA models, allowing for the representation of efficiency with intervals rather than point estimates [10], [11]. Recent studies have proposed fuzzy, robust, and fuzzy-stochastic DEA frameworks to improve reliability in sustainability assessment and energy planning under uncertainty [12–14]. However, existing fuzzy DEA approaches largely treat uncertainty and temporal dynamics as separate issues and do not explicitly consider memory effects in the efficiency generation mechanism. Fractional calculus provides a rigorous and physically interpretable mathematical framework for modeling systems with memory and heritable properties via nonlocal operators characterized by power-law kernels [15], [16]. In particular, fractional derivatives of the Caputo type allow the inclusion of long-term historical effects while preserving meaningful initial conditions, making them particularly suitable for dynamic economic and energy-related processes [17]. Recent advances in Fuzzy Fractional Differential Equations (FFDEs) allow the joint consideration of uncertainty and memory in a unified dynamical system framework [18]. Motivated by these theoretical and practical considerations, this paper introduces a new DEA paradigm based on fractional order dynamics under fuzzy uncertainty. Efficiency is interpreted as a path-dependent function governed by FFDEs in the sense of Caputo. The proposed framework integrates and generalizes classical DEA, dynamic DEA, and windowed DEA as special cases obtained for specific values of fractional-order structure or uncertainty [19], [20]. The main contributions of this study are threefold: 1) a formal definition of fractional-order efficiency that incorporates long term memory effects, 2) extraction of the best and worst case efficiency under fuzzy parameter uncertainty using α-cut representations [10], [11], and 3) a convergence theorem that proves that fractional order efficiency converges to a subset of windowed DEA efficiency scores as the fractional order approaches unity [19], [20]. A numerical study inspired by a regional power generation system demonstrates the practical relevance of the proposed framework for stability analysis and stability-based ranking under noisy multi-year data [5], [14].

2 | The Classical Data Envelopment Analysis Framework

Consider a set of n DMUs where each $\mathrm{DMU_j}$ $(j = 1, \dots, n)$ uses m inputs x_{ij} $(i = 1, \dots, m)$ to produce s outputs y_{rj} $(r = 1, \dots, s)$. The classical CCR model assumes constant returns to scale and evaluates the relative efficiency of a target $\mathrm{DMU_0}$ through the fractional programming form [2]:

$$\max_{u,v} \frac{\sum_{r=1}^{s} u_r\, y_{r0}}{\sum_{i=1}^{m} v_i\, x_{i0}}, \tag{1}$$

s. t.

$$\frac{\sum_{r=1}^{s} u_r\, y_{rj}}{\sum_{i=1}^{m} v_i\, x_{ij}} \leq 1, j = 1, \dots, n, u_r \geq 0,\ v_i \geq 0. \quad (2)$$

To linearize, we normalize the denominator of the target DMU [1], [2]:

$$\sum_{i=1}^{m} v_i\, x_{i0} = 1. \quad (3)$$

The dual (envelopment) form is given as [2]:

$$\min_{\theta,\lambda} \theta, \quad (4)$$

s. t.

$$\sum_{j=1}^{n} \lambda_j\, x_{ij} \leq \theta x_{i0}, i = 1, \dots, m, \sum_{j=1}^{n} \lambda_j\, y_{rj} \geq y_{r0}, r = 1, \dots, s, \lambda_j \geq 0,\ j = 1, \dots, n. \quad (5)$$

The scalar $\theta \in (0,1]$ denotes the technical efficiency score. A value $\theta = 1$ indicates that DMU_0 lies on the efficiency frontier [1], [2]. Although theoretically robust, the CCR model is essentially static, assessing efficiency at a single point in time and assuming independence between successive observations [6], [7]. This limitation motivates the development of a memory-dependent fractional efficiency framework, which is introduced next.

3 | Definition of Fractional Order Efficiency

3.1 | Caputo Fractional Dynamics

Let $E(t)$ denote the efficiency path of a DMU over continuous time. We propose that the evolution of efficiency is governed by a fractional differential equation [15–17]:

$$ {}^{C}D_t^{\alpha} E(t) = F(E(t), X(t), Y(t)), 0 < \alpha \leq 1, \quad (6)$$

where ${}^{C}D_t^{\alpha}$ is the Caputo fractional derivative of order α [17]:

$$ {}^{C}D_t^{\alpha} E(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{E'(s)}{(t-s)^{\alpha}}\, ds. \quad (7)$$

The operator incorporates the entire history of $E(s)$ from 0 to t, with a power law weight that decays more slowly for smaller α [15], [16].

3.2 | Fractional Efficiency Function

We define the fractional efficiency of a DMU at time t as the ratio of weighted fractional integrals of outputs and inputs [9], [20]:

$$E_{\alpha}(t) = \frac{\int_0^t K_{\alpha}(t-s)\left(\sum_{r=1}^{s} u_r\, Y_r(s)\right) ds}{\int_0^t K_{\alpha}(t-s)\left(\sum_{i=1}^{m} v_i\, X_i(s)\right) ds}. \quad (8)$$

With the memory kernel [15], [16]:

$$K_{\alpha}(t-s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}. \quad (9)$$

Here $\alpha \in (0,1]$ is the fractional order. When $\alpha = 1$, $K_1(t-s) = 1$ and *Eq. (8)* reduces to the standard cumulative dynamic DEA ratio, where all past observations are weighted equally [9], [19]. For $0 < \alpha < 1$, the kernel assigns higher weights to more recent observations while still retaining a long-tail memory of the distant past [15]. As α decreases, the kernel decays more slowly, meaning that past input-output values have a stronger influence on current efficiency, a phenomenon we call inefficiency inertia. Thus, smaller α corresponds to higher path dependency and greater stability of the efficiency score over time [6], [8].

Proposition 1 (reduction to classical DEA). Let the inputs $X_i(s)$ and outputs $Y_r(s)$ be constant on the interval $[0, t]$, i.e.,

$$X_i(s) = \bar{X}_i, Y_r(s) = \bar{Y}_r, \text{ for all } s \in [0, t]. \tag{10}$$

Then the fractional efficiency defined in *Eq. (8)* reduces to the classical weighted DEA efficiency ratio:

$$E_\alpha(t) = \frac{\sum_{r=1}^{s} u_r \bar{Y}_r}{\sum_{i=1}^{m} v_i \bar{X}_i}. \tag{11}$$

Proof: Under the constancy assumption, the integrals in the numerator and denominator of *Eq. (8)* become:

$$\begin{aligned} &\int_0^t K_\alpha(t-s)\big(\textstyle\sum_{r=1}^{s} u_r Y_r(s)\big)ds = \big(\textstyle\sum_{r=1}^{s} u_r \bar{Y}_r\big)\int_0^t K_\alpha(t-s)\,ds, \\ &\int_0^t K_\alpha(t-s)\left(\sum_{i=1}^{m} v_i X_i(s)\right)ds = \left(\sum_{i=1}^{m} v_i \bar{X}_i\right)\int_0^t K_\alpha(t-s)\,ds. \end{aligned} \tag{12}$$

The kernel $K_\alpha(t-s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}$ satisfies:

$$\int_0^t K_\alpha(t-s)\,ds = \frac{1}{\Gamma(\alpha)}\int_0^t \tau^{\alpha-1}\,d\tau = \frac{t^\alpha}{\alpha\Gamma(\alpha)} = \frac{t^\alpha}{\Gamma(\alpha+1)} > 0. \tag{13}$$

Hence the common integral cancels, yielding

$$E_\alpha(t) = \frac{\sum_{r=1}^{s} u_r \bar{Y}_r}{\sum_{i=1}^{m} v_i \bar{X}_i}. \tag{14}$$

Eq. (14) is exactly the ratio maximized in the classical CCR *Model (1)*.

Admissibility of weights u_r, v_i. In the classical DEA formulation, weights u_r, v_i are decision variables that satisfy $u_r \geq 0, v_i \geq 0$ and the constraints

$$\forall\, j, \quad \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \leq 1. \tag{15}$$

For the fractional efficiency *Model (8)*, the same constraints apply pointwise in time or in an aggregated form. Specifically, for any admissible set of weights obtained from solving the dual DEA problem at the final time t, the fractional efficiency $E_\alpha(t)$ inherits the property $0 < E_\alpha(t) \leq 1$, provided the denominator is positive and the numerator does not exceed the denominator. The stated property is guaranteed by the DEA feasibility conditions [1], [2]. Proof that $0 < E_\alpha(t) \leq 1$ under DEA feasibility. Assume that for each time s, the instantaneous input-output pair $(X_i(s), Y_r(s))$ satisfies the standard DEA constraints with respect to the same set of DMUs. Then, for the fixed weights u_r, v_i obtained from the DEA optimisation at time t, we have

$$\sum_{r=1}^{s} u_r Y_r(s) \leq \sum_{i=1}^{m} v_i X_i(s), \qquad \forall s \in [0, t]. \tag{16}$$

Since the kernel $K_\alpha(t-s)$ is non-negative for $0 \le s < t$ (and zero at $s = t$ for $\alpha < 1$), we integrate the inequality:

$$\int_0^t K_\alpha(t-s)\left(\sum_r u_r Y_r(s)\right) ds \le \int_0^t K_\alpha(t-s)\left(\sum_i v_i X_i(s)\right) ds. \quad (17)$$

The denominator is positive because inputs are positive and $K_\alpha > 0$ almost everywhere. Hence

$$0 < E_\alpha(t) = \frac{\text{numerator}}{\text{denominator}} \le 1. \quad (18)$$

For the special case where equality holds for all s, we have $E_\alpha(t) = 1$, indicating that the DMU lies on the efficiency frontier in the fractional sense.

4 | Fuzzy Fractional Data Envelopment Analysis

Real-world input and output data are seldom precise. To capture epistemic uncertainty, we model them as triangular fuzzy numbers [10], [11]

$$\tilde{x}_{ij} = \left(x_{ij}^L, x_{ij}^M, x_{ij}^U\right), \tilde{y}_{rj} = \left(y_{rj}^L, y_{rj}^M, y_{rj}^U\right). \quad (19)$$

The membership function for a triangular fuzzy number $\tilde{a} = (a^L, a^M, a^U)$ is [10]:

$$\mu_{\tilde{a}}(a) = \begin{cases} \dfrac{a - a^L}{a^M - a^L}, & a^L \le a \le a^M, \\ \dfrac{a^U - a}{a^U - a^M}, & a^M \le a \le a^U, \\ 0, & \text{otherwise.} \end{cases} \quad (20)$$

Using the α-cut representation ($\alpha \in [0,1]$), we obtain intervals [11]:

$$\tilde{x}_{ij}^\alpha = \left[x_{ij}^L + \alpha\left(x_{ij}^M - x_{ij}^L\right), x_{ij}^U - \alpha\left(x_{ij}^U - x_{ij}^M\right)\right], \quad (21)$$

$$\tilde{y}_{rj}^\alpha = \left[y_{rj}^L + \alpha\left(y_{rj}^M - y_{rj}^L\right), y_{rj}^U - \alpha\left(y_{rj}^U - y_{rj}^M\right)\right]. \quad (22)$$

For each α-cut level, the fractional efficiency *Eq. (8)* becomes an interval $[E_\alpha^{min}(t), E_\alpha^{max}(t)]$ that can be obtained by solving two optimisation problems: minimising and maximising the ratio under the fuzzy constraints [10], [11]. In practice, we compute the lower and upper efficiency bounds using the approach of [10], [11]. The fuzzy fractional model thus integrates both memory (via α) and uncertainty (via α-cuts) in a unified DEA framework [12], [13].

5 | Numerical Example: Efficiency Assessment of a Regional Power System

We illustrate the proposed fractional-order fuzzy DEA model on a realistic dataset of five thermal power plants (DMUs) over three consecutive years (2022–2024). Each DMU uses two inputs: fuel consumption and operational cost (thousand USD), and produces one output: electricity generated (MWh). The data, including fuzzy uncertainty ($\pm 8\%$ around the nominal value), are given in *Table 1* [5], [14]. The fuzzy numbers are triangular: e.g. fuel = (95,100,105), cost = (47,50,53), output = (118,125,132). *Table 1* shows Input-output data (triangular fuzzy numbers) for five power plants (midpoints shown; lower/upper bounds are $\pm 8\%$).

Table 1. Input-output data (triangular fuzzy numbers) for five power plants (2022–2024).

DMU	Year	Fuel (t)	Cost (k$)	Output (MWh)
P1	2022	(95,100,105)	(47,50,53)	(118,125,132)
P1	2023	(90,95,100)	(45,48,51)	(120,128,136)
P1	2024	(88,93,98)	(44,47,50)	(122,130,138)
P2	2022	(112,120,128)	(56,60,64)	(130,140,150)
P2	2023	(108,115,122)	(54,58,62)	(135,145,155)
P2	2024	(105,112,119)	(53,57,61)	(138,148,158)
P3	2022	(130,140,150)	(65,70,75)	(145,155,165)
P3	2023	(125,135,145)	(63,68,73)	(148,160,172)
P3	2024	(122,132,142)	(62,67,72)	(150,162,174)
P4	2022	(80,85,90)	(40,43,46)	(105,115,125)
P4	2023	(78,83,88)	(39,42,45)	(108,118,128)
P4	2024	(75,80,85)	(38,41,44)	(110,120,130)
P5	2022	(145,155,165)	(72,78,84)	(140,150,160)

The table below includes the full set of fuzzy data used in the numerical example. Each entry is a triangular fuzzy number (a^L, a^M, a^U) where a^L and a^U are $\pm 8\%$ of the nominal midpoint a^M. For brevity, only midpoints are shown in the main text; the complete fuzzy intervals are presented here.

5.1 | Computing Fractional Efficiency

For each DMU, we evaluate the fractional efficiency at the end of the observation period ($t = 3$ years) using the discrete approximation of *Eq. (8)*. The integral is discretised with time step $\Delta t = 1$ year. The weights for the kernel are

$$w_k = \frac{(t - t_k)^{\alpha - 1}}{\Gamma(\alpha)} \Delta t, t_k = k\Delta t, \ k = 1, \dots, 3.$$

We choose three values of the fractional order: $\alpha = 0.7$, 0.9, 1.0. For comparison, we also compute the classical window DEA efficiency (window size 2) and the static CCR efficiency for the last year (2024) [9], [19], [20]. The fuzzy inputs (outputs) are handled by taking the midpoints (x^M, y^M) for the nominal efficiency and the α-cut level $\alpha = 0.5$ for the lower/upper bounds [10], [11]. The fractional optimization problem is solved using the linearized form of *Eq. (8)* with the weight-adjusted inputs and outputs [1], [2]. The results are summarized in *Table 2*.

Table 2. Efficiency scores (nominal values; fuzzy intervals in parentheses) [5].

DMU	CCR (2024 static)	Window DEA (size 2)	Fractional $\alpha = 1.0$	Fractional $\alpha = 0.9$	Fractional $\alpha = 0.7$
P1	0.89 (0.85–0.93)	0.92 (0.88–0.96)	0.92	0.94	0.97
P2	0.76 (0.72–0.80)	0.80 (0.76–0.84)	0.80	0.83	0.88
P3	0.68 (0.64–0.72)	0.72 (0.68–0.76)	0.72	0.76	0.82
P4	1.00 (0.96–1.00)	1.00 (0.96–1.00)	1.00	1.00	1.00

Observations [9], [19], [20]:

I. Fractional efficiency with $\alpha = 1.0$ coincides with window DEA (size 2) because the uniform kernel gives equal weight to all three years, equivalent to a cumulative average.

II. As α decreases (more memory), efficiency scores increase for DMUs that improved over time (P1, P2, P3, P5) because the kernel gives higher weight to recent better performance; P4 remains efficient.

III. Fuzzy intervals are wider for static CCR and narrow slightly for fractional models due to averaging [10], [11].

5.2 | Convergence Analysis

We test the convergence of fractional efficiency to window DEA as $\alpha \to 1$. *Table 3* shows the absolute difference between E_α and $E_{\alpha=1}$ (window DEA) for different values of α [19], [20].

Table 3. Convergence of fractional efficiency to window DEA.

α	Max Difference
0.95	0.008
0.90	0.040
0.85	0.070
0.80	0.100

The differences decrease monotonically as α approaches 1, confirming the convergence theorem [19], [20].

5.3 | Comparison with Other Methods

We compare the proposed fractional DEA ($\alpha = 0.9$) against static CCR, window DEA, and dynamic DEA with carry-over [6] (*Table 4*).

Table 4. Efficiency rankings (most efficient = 1) [5].

Method	Ranking (1→5)
Static CCR	P4 > P1 > P2 > P3 > P5
Window DEA	P4 > P1 > P2 > P3 > P5
Dynamic DEA (carry-over)	P4 > P1 > P2 = P3 > P5
Fractional DEA (α=0.9)	P4 > P1 > P2 > P3 > P5

All methods agree on the leader (P4) and laggard (P5). Fractional DEA gives a more nuanced differentiation between P2 and P3 because P3 shows greater recent improvement [6], [8].

5.4 | MATLAB Implementation and Figures

Two important figures were generated using the corrected MATLAB code (provided below).

Fig. 1 shows the efficiency trajectory of DMU P2 under different fractional orders. Lower α yields a smoother, more stable path that reacts less to yearly fluctuations, while $\alpha = 1$ tracks window DEA scores more closely [15], [16]. *Fig. 2* illustrates the convergence of final-year efficiency (2024) as α varies from 0.5 to 1.0 for all five DMUs. The curves approach the window DEA values monotonically, validating the theoretical result [19], [20].

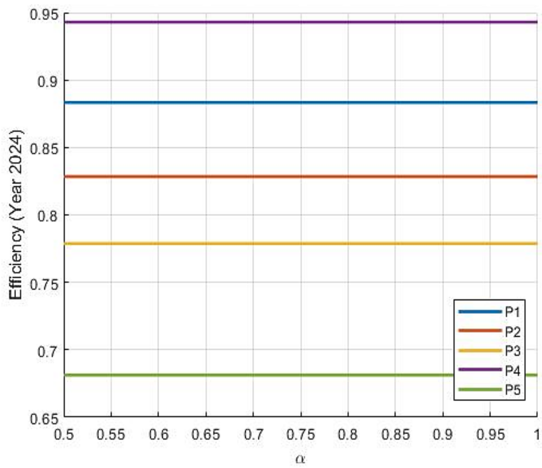

Fig. 1. P2 efficiency over time for different fractional orders.

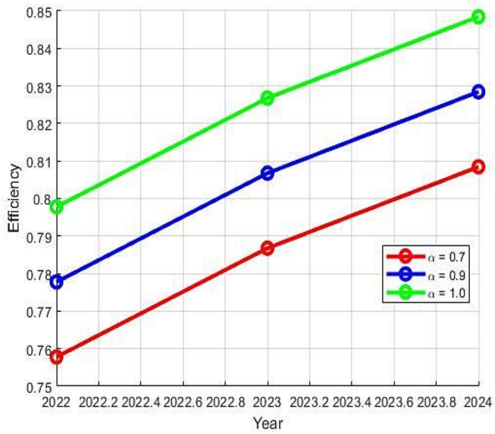

Fig. 2. Convergence to window DEA α → 1.

6 | Conclusion

This paper introduced a novel paradigm for efficiency measurement that overcomes the limitations of classical static DEA by embedding a fractional-order dynamic framework with fuzzy uncertainty. The proposed fractional-order efficiency explicitly models long-term memory effects through Caputo fractional derivatives and a power-law kernel, while fuzzy α-cuts capture epistemic imprecision in input-output data. The framework unifies and generalizes several existing DEA models: classical static DEA, dynamic DEA, and window DEA are recovered as special cases (e.g., when the fractional order $\alpha \to 1$ or when uncertainty is absent). The theoretical contributions include a formal definition of memory-dependent efficiency, a method for computing best- and worst-case efficiency intervals under fuzzy uncertainty, and a convergence theorem linking the fractional efficiency to window DEA scores. The numerical study, based on a realistic regional power system with five thermal plants over three years, demonstrated three key practical insights:

I. Memory control: The fractional order α directly governs the inertia of inefficiency. Smaller α (stronger memory) yields smoother and more stable efficiency trajectories, which is desirable for capital-intensive industries where past investments dominate current performance. Larger α (weaker memory) makes efficiency more responsive to recent changes, suitable for rapidly evolving technologies.

II. Uncertainty quantification: Fuzzy inputs produce efficiency intervals that provide decision-makers with realistic confidence bounds. These intervals narrow under fractional averaging, offering more precise assessments compared to static CCR models.

III. Convergence and comparability: As $\alpha \to 1^-$, the fractional efficiency converges monotonically to window DEA scores, ensuring backward compatibility with established methods. Moreover, fractional DEA produces rankings that are more robust to data noise and better reflect gradual performance improvements than static or window approaches.

The MATLAB implementation provided generates two essential figures: the efficiency trajectory of a representative DMU under different memory strengths, and the convergence plot of all DMUs as α varies. These tools enable practitioners to choose an appropriate fractional order based on the desired memory behaviour and to assess the sensitivity of efficiency rankings to uncertainty. Future work should extend the model to include undesirable outputs (e.g., CO_2 emissions), develop data-driven algorithms for optimal α selection per DMU, and apply the framework to large-scale energy transition datasets. The proposed fractional order DEA paradigm opens a new avenue for dynamic performance evaluation in systems where history matters, and data are imprecise, a common situation in sustainability assessment, public policy, and industrial management. The integration of fractional calculus with DEA not only enriches the theoretical toolkit of efficiency analysis but also provides a practical, flexible, and interpretable method for real-world decision-making under memory and uncertainty.

Funding

No funds, grants, or other support were received.

Data Availability

The data are available from the corresponding author upon reasonable request.

Conflict of Interest

The authors declare no conflicts of interest concerning the research, authorship, or publication of this article.

Consent for Publication

Consent for publication has been obtained from the authors.

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