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Radial Models for Classifying Flexible Measures in Two-Stage Network DEA-RA

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Abstract

In conventional Data Envelopment Analysis (DEA) models it has been assumed that each measure status is considered input or output. However, a performance measure in some cases can have input role for some DMUs and output role for others and is known as flexible measure. In this paper new radial FNDEA-R models are proposed in the presence of flexible measures based on the ratio of input components to output components or vice versa in the input and output orientation under constant returns to scale in general two-stage network. In our proposed models, flexible measures are determined as input or output to improve performance to maximize the relative efficiency of the DMU under evaluation. The FNDEA-R models versus FNDEA models prevent efficiency underestimation and pseudo inefficiency issues. The status of one flexible measure in the input-oriented and output-oriented FNDEA-R models may have different conclusions. The radial FNDEA and FNDEA-R models have units-invariant. A numerical example is used to illustrate the procedures.

Keywords: Data envelopment analysis, Flexible measures, Ratio analysis, Radial models, General two-stage network.

1 | Introduction

Data Envelopment Analysis (DEA) has demonstrated to be an effective technique for measuring the relative efficiency of a set of homogeneous DMUs which utilize the same inputs to produce the same outputs. In conventional DEA applications, given a set of available measures, it is assumed that the status of each measure is clearly stated as an input or an output variable in the production process prior to using DEA. However, in some situations, a performance measure can play input role for some DMUs and output role for others. In Cook and Zhu [1] flexible variables were used and they introduced an adjustment to the standard constant returns to scale DEA model to classify these variables [2].

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According to Amirteimoori and Emrouznejad [3] and Toloo [4] argued that a disadvantage in the method is its need to have extra information to make decision about each variable's role. In addition, with alternative optimal solutions in the models, the outcomes for selected flexible measure (either input or output) are identical for some DMUs and it is reasonable not to take them into account for input and output classification. An SORM model was introduced by Kordrostami [5] capable of evaluating the efficiency of units with negative and flexible data. Tohidi and Matroud [6] introduced a novel non-oriented model for determining each flexible measure status as an input or output. In addition, they proposed the aggregated model and an extension with negative data pertinent to the proposed approach.

Tavana et al. [7] introduced a non-radial directional distance model for classification of flexible measures. Jahani Sayyad Noveiri et al. [8] introduced an FSBM model with integer-value to assess the relative efficiency of DMUs in which flexible and integer measures were available. Novel radial models were proposed by Hosseini Monfared et al. [9] to classify flexible measures, which used envelopment form of CCR with constant and variable returns to scale in basic two-stage network structure. They also proposed integer-valued FNDEA methods capable of evaluating the relative efficiency of DMUs and determine the flexile measure status with integer data in general and basic two-stage network structure. Hosseini Monfared et al. [10] proposed FNDEA models using a multiplier model with variable and constant returns to scale (CRS & VRS) with general two-stage network structure. The model was employed to classify flexible measures where each one was treated as output or input to maximize overall network efficiency of the DMU under evaluation.

In several cases, it is the intention of managers to employ inputs/outputs ratios instead of original data. For instance, the ratio of discharged patients to the total number of patients in a hospital. To examine the efficiency of a group of DMUs with ratio data, DEA-R models have a better performance than traditional models. In practice, there was several cases where data is represented as ratio and input/output or output/input data are essential for decision making. In some cases, the data is given as ratio or percentage data. Generally, ratio data can be categorized in three categories. First category is when some output and input elements are ratio and the rest are as volume. The numerator and denominator representing such data are available and the ratio data can be added to the model as a decimal number. The axiom convexity of the assumptions used to estimate the production possibility set is not determined with given ratio data in this category.

Hatami-Marbini and Toloo [11] introduced DEA models for ratio data and solved the drawbacks in Emrouznejad and Amin's [12] models. DEA models were proposed by Emrouznejad and Amin [12] for ratio data through changing the convexity axiom of the key assumption to estimate Production Possible Set (PPS). The 2nd category contains input and output elements as ratio and the rest of elements are as volume. The category includes numerator and denominator corresponding to the data which are not available and ratio data is represent as a decimal number. In addition, data ratio is employed as a decimal number. The convexity principle is not determined with the presence of ratio data. Olesen et al. [13] used this category and introduced new DEA models to measure efficiency with given ratio and volume data. The third category includes models which are obtained through mixing ratio analysis models and DEA. Ratio data in this category are not ratio inherently and they are obtained through dividing input elements to output element or vice versa.

In addition, the denominator and numerators of these data are given as input or output elements. Models like this are known as DEA-Ratio based (DEA-R) models where the output and input elements are given and directly added to the model. All the ratios corresponding to the output and input elements are employed to assess efficiency. A DEA-R model was introduced by Despic et al. [14] through mixing ratio analysis, DEA models and DEA-R models in the output orientation to obtain the DMUs efficiency when ratio data is available. Wei et al. [15] studied DEA-R models in input orientation and reported that with DEA-R models, the possible problems of standard DEA models are resolved. Mozaffari et al. [16], [17] utilized DEA-R models to assess revenue and cost efficiency. They introduced a relationship in DEA models without explicit input and DEA-R models. Gerami et al. [18] proposed DEA-R models for measuring two-stage network structures efficiency with ratio data. Gerami et al. [19] developed the SBM model for ratio data using DEA-R models.

Their model demonstrated super-efficiency and efficiency scores with corresponding slack values for output/input and input/output ratio based on the orientation of the production frontier. Ghiyasi et al. [20] extended the theoretical basis of the inverse DEA-R model as a post-efficiency analysis manner.

The presentation of models in the presence of flexible measures based on the ratio of input/output ratios or vice versa is critical for decision maker in two-stage network structures. Our goals in this research are to use radial models in DEA and DEA-R for classifying flexible measures. In proposed models in DEA-R flexible measures are determined as input or output in the numerator or denominator of ratios to increase relative efficiency. By solving a model based on the radial model in DEA and DEA-R, both flexible measures are classified and the relative efficiency of the units is measured. Novel radial FNDEA-R models are proposed in the presence of flexible measures for the third category of ratio data using the input/output ratio and vice versa in the input and output orientation with constant returns to scale in general two-stage network DEA-R to classify flexible measures where each flexible measure has an input or output role to achieve the highest relative efficiency of DMU under evaluation.

The models are good with key disadvantages in DEA models such as inaccurate estimates of efficiency, weak efficiency and pseudo-inefficiency. The efficiency scores obtained from the FNDEA-R models are greater than or equal to the efficiency scores obtained from the FNDEA model. The FNDEA-R models provide a bigger feasible space for selecting the corresponding weight of output/input ratios and input/output ratios. Therefore, all the input-to-output ratios or vice versa are included to measure the DMU efficiency. The condition of the flexible measure in the input-oriented and output-oriented FNDEA-R models in the general two-stage network structure may have different conclusions and it is expectable to select a flexible measure as an input measure for one model while an output for another. In proposed models in DEA-R flexible measures are determined as input or output in the numerator or denominator of ratios to increase relative efficiency. By solving a model based on the radial model in DEA and DEA-R, both flexible measures are classified and the relative efficiency of the units is measured. In the radial models in DEA-R, all inputs and outputs should be positive and the ratios of inputs to outputs and vice versa should be defined. In the proposed radial models in DEA and DEA-R, constraints are linear, although there are binary variables in these models. The proposed radial models in DEA and DEA-R are always feasible and bounded.

The paper is organized as follows. In the second section, the previous studies on classifying flexible measures and DEA-R models are reviewed in brief. In the third section, new radial FNDEA and FNDEA-R models are proposed with flexible measures based on input/output ratio or vice versa in the input and output orientation under constant returns to scale in general two-stage network to classify flexible measures to maximize the DMU relative efficiency. In the fourth section, A numerical example is employed for explaining more details while the Section 5 represents the conclusion.

2 | Preliminaries

In this section, we review the previous studies on DEA-R models in brief.

2.1 | Ratio-Based DEA Models

Assume the purpose is to evaluate the efficiency of n decision making units $(\mathrm{DMU}_j, j = 1, \dots, n)$ in which there are m inputs $x_{ij} > 0, i = 1, \dots, m$, s outputs $y_{rj} > 0, r = 1, \dots, s$.

2.1.1 | Input-oriented envelopment DEA-R model (DEA-R-I)

Suppose we have n decision units as $(\mathrm{DMU}_j, j = 1, \dots, n)$. Each DMU uses the input vector $x_j = (x_{1j}, \dots, x_{mj})$ to generate the output vector $y_j = (y_{1j}, \dots, y_{sj})$. Wei et al. [15] presented the input-oriented envelopment DEA-R model (DEA-R-I) based on the ratio of inputs to outputs and the output-oriented envelopment DEA-R model (DEA-R-O) model based on the ratio of outputs to inputs as follows:

$$\theta^* = \min \theta,$$
$$\text{s.t.} \quad \sum_{j=1}^{n} \lambda_j \left(\frac{x_{ij}}{y_{rj}} \right) \leq \theta \left(\frac{x_{io}}{y_{ro}} \right), \quad i = 1, \ldots, m, \quad r = 1, \ldots, s, \tag{1}$$
$$\sum_{j=1}^{n} \lambda_j = 1, \quad \lambda_j \geq 0, \; j = 1, \ldots, n. \quad \theta, \quad \text{free in sign.}$$

Model (1) is a DEA-R radial model in input orientation and provides the amount of efficiency based on the ratio of inputs to outputs in CRS technology.

Definition 1. $\mathrm{DMU_o}$ is DEA-R efficient unit with input orientation if and only if $\theta^* = 1$.

2.1.2 | Output-oriented envelopment DEA-R model (DEA-R-O)

Model (2) is a DEA-R radial model in output orientation (DEA-R-O) based on the ratio of outputs to inputs as follows:

$$\varphi^* = \max \varphi,$$
$$\text{s.t.} \quad \sum_{j=1}^{n} \lambda_j \left(\frac{y_{rj}}{x_{ij}} \right) \geq \varphi \left(\frac{y_{ro}}{x_{io}} \right), \quad i = 1, \ldots, m, \quad r = 1, \ldots, s, \tag{2}$$
$$\sum_{j=1}^{n} \lambda_j = 1, \quad \lambda_j \geq 0, \; j = 1, \ldots, n. \quad \varphi \geq 0.$$

This model provides the amount of efficiency based on the ratio of outputs to inputs in CRS technology.

Definition 2. $\mathrm{DMU_o}$ is DEA-R efficient unit with output orientation if and only if $\varphi^* = 1$.

3 | New Radial Models in DEA and DEA-RA

New radial FNDEA-R models are proposed with flexible measures using the input/output ratio or vice versa in the input and output orientation under constant returns to scale in general two-stage network to classify flexible measures to achieve the highest relative efficiency of the DMU under evaluation. A general two-stage network structure is pictured in *Fig. 1*. Each $\mathrm{DMU_j}, (j = 1, \ldots, n)$ contains m inputs $x_{ij}, (i = 1, \ldots, m)$ to the stage one and H outputs $y'_{hj}, (h = 1, \ldots, H)$ leaving as outputs. Along with the H outputs, the stage one contains K outputs $z_{kj}, (k = 1, \ldots, K)$ known as intermediate measures or links that become inputs to the second stage. The second stage has its own inputs $x'_{tj}, (t = 1, \ldots, T)$. The outputs from the second stage are $y_{rj}, (r = 1, \ldots, s)$.

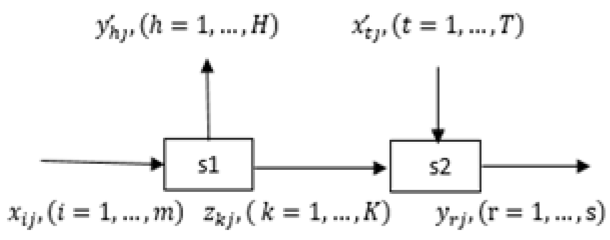

Fig. 1. General two-stage network.

If there are L_1 flexible measures to stage one $w^1_{l_1 j}$, $(l_1 = 1, \ldots, L_1)$ and L_2 flexible measures to the second stage $w^2_{l_2 j}$, $(l_2 = 1, \ldots, L_2)$ of which the statues of input or output are not given, some DMUs can employ these measures as inputs while other DMUs can be employed as outputs.

3.1 | Radial Model for Classifying Flexible Measures in DEA

A new radial FNDEA model is presented using envelopment form of CCR in the input orientation in general two-stage network DEA to assess the relative efficiency and categorize flexible measures.

$$
\begin{aligned}
&\theta^*_{FNDEA} = \min \quad \theta, \\
&\text{s.t.} \quad \sum_{j=1}^{n} \lambda_j^1 x_{ij} \le \theta x_{io}, && i = 1,\dots,m, \\
&\sum_{j=1}^{n} \lambda_j^1 w^1_{l_1 j} \le \theta w^1_{l_1 o} + M d^1_{l_1}, && l_1 = 1,\dots,L_1, \\
&\sum_{j=1}^{n} \lambda_j^1 d^1_{l_1} w^1_{l_1 j} \ge \sum_{j=1}^{n} \lambda_j^2 d^1_{l_1} w^1_{l_1 j}, && l_1 = 1,\dots,L_1, \\
&\sum_{j=1}^{n} \lambda_j^1 y'_{hj} \ge y'_{ho}, && h = 1,\dots,H, \\
&\sum_{j=1}^{n} \lambda_j^2 x'_{tj} \le \theta x'_{to}, && t = 1,\dots,T, \\
&\sum_{j=1}^{n} \lambda_j^2 y_{rj} \ge y_{ro}, && r = 1,\dots,s\ , \\
&\sum_{j=1}^{n} \lambda_j^2 w^2_{l_2 j} \le \theta w^2_{l_2 o} + M d^2_{l_2}, && l_2 = 1,\dots,L_2, \\
&\sum_{j=1}^{n} \lambda_j^2 w^2_{l_2 j} \ge w^2_{l_2 o} - M\left(1 - d^2_{l_2}\right), && l_2 = 1,\dots,L_2, \\
&\sum_{j=1}^{n} \lambda_j^1 z_{kj} \ge \sum_{j=1}^{n} \lambda_j^2 z_{kj}, && k = 1,\dots,K, \\
&d^1_{l_1} \in \{0,1\},\ d^2_{l_2} \in \{0,1\}, && \text{for all } l_1, l_2, \\
&\lambda_j^1 \ge 0,\ \lambda_j^2 \ge 0, && j = 1,\dots,n.
\end{aligned}
\qquad (3)
$$

In this model, M is a big positive number. For each l_1 and l_2, binary variables $d^{(1)}_{l_1}$ and $d^{(2)}_{l_2} \in \{0,1\}$ to the first and second stage are used for identifying that factors l_1 and l_2 are used as inputs to the stages one and two, $d^{(1)}_{l_1} = 0$ and $d^{(2)}_{l_2} = 0$ or outputs of stages one and two $d^{(1)}_{l_1} = 1$ and $d^{(2)}_{l_2} = 1$.

Definition 3. DMU_o is called a FNDEA efficient unit if and only if $\theta^*_{FNDEA} = 1$.

The technology set T_{FNDEA} is defined as follows:

$$
T_{FNDEA} = \left\{ (x, w^1, y', z, x', w^2, y) \;\middle|\; \begin{array}{c} \sum_{j=1}^{n} \lambda_j^1 x_j \le x; \sum_{j=1}^{n} \lambda_j^1 y'_j \ge y'; \begin{pmatrix} \sum_{j=1}^{n} \lambda_j^1 w_j^1 \le w^1 \text{or} \\ \sum_{j=1}^{n} \lambda_j^1 w_j^1 \ge \sum_{j=1}^{n} \lambda_j^2 w_j^1 \end{pmatrix}; \\ \sum_{j=1}^{n} \lambda_j^1 z_j \ge \sum_{j=1}^{n} \lambda_j^2 z_j; \sum_{j=1}^{n} \lambda_j^2 x'_j \le x'; \sum_{j=1}^{n} \lambda_j^2 y_j \ge y; \begin{pmatrix} \sum_{j=1}^{n} \lambda_j^2 w_j^2 \le w^2 \text{or} \\ \sum_{j=1}^{n} \lambda_j^2 w_j^2 \ge w^2 \end{pmatrix}; \lambda_j^1 \ge 0, \lambda_j^2 \ge 0, \text{for all } j \end{array} \right\}.
$$

Definition 4. *Model (3)* is feasible and has units-invariant.

3.2 | Classifying Flexible Measures in DEA-RA

Here new radial FNDEA-R models are presented in the presence of flexible measures under constant returns to scale in the output and input orientation depending on input/output ratios and vice versa where each flexible measure functions as input for some DMUs and output for the others to achieve a maximum relative efficiency of the DMU under evaluation in general two-stage network.

3.2.1 | Input-oriented radial FNDEA-R-I model

A general two-stage network structure is illustrated in *Fig. 1*. Each $DMU_j, (j = 1, \ldots, n)$ contains m inputs $x_{ij} > 0, (i = 1, \ldots, m)$ at the first stage and H outputs $y'_{hj} > 0, (h = 1, \ldots, H)$ leaving the system. Along with H outputs, the first stage contains K outputs $z_{kj} > 0, (k = 1, \ldots, K)$ known as intermediate measures or links that function as inputs for the second stage. The second stage has its inputs $x'_{tj} > 0, (t = 1, \ldots, T)$. The second stage outputs are $y_{rj} > 0, (r = 1, \ldots, s)$. In addition, assume L_1 flexible measures to the first stage $w^1_{l_1j} > 0$, $(l_1 = 1, \ldots, L_1)$ and L_2 flexible measures to the second stage $w^2_{l_2j} > 0$, $(l_2 = 1, \ldots, L_2)$ in which the statuses of input or output is not given. Some of these measures are used as input by some DMUs and the rest are used as output by other DMUs. We define division data sets, which are $(m \times H), (L_1 \times H), (m \times L_1), (L_1 \times s), (T \times s), (L_2 \times s), (T \times L_2), (m \times K)$ and $(K \times s)$ dimensions vectors as given next:

$$\left\{\frac{x}{\acute{y}} = \left(\frac{x_1}{\acute{y}_1}, \ldots, \frac{x_m}{\acute{y}_1}, \frac{x_1}{\acute{y}_2}, \ldots, \frac{x_m}{\acute{y}_2}, \ldots, \frac{x_1}{\acute{y}_H}, \ldots, \frac{x_m}{\acute{y}_H}\right)\right\} \text{ with } \acute{y} = (\acute{y}_1, \ldots, \acute{y}_H),\ x = (x_1, \ldots, x_m).$$

$$\left\{\frac{w^1}{\acute{y}} = \left(\frac{w^1_1}{\acute{y}_1}, \ldots, \frac{w^1_{L_1}}{\acute{y}_1}, \frac{w^1_1}{\acute{y}_2}, \ldots, \frac{w^1_{L_1}}{\acute{y}_2}, \ldots, \frac{w^1_1}{\acute{y}_H}, \ldots, \frac{w^1_{L_1}}{\acute{y}_H}\right)\right\} \text{ with } \acute{y} = (\acute{y}_1, \ldots, \acute{y}_H),\ w^1 = \left(w^1_1, \ldots, w^1_{L_1}\right).$$

$$\left\{\frac{x}{\acute{w}^1} = \left(\frac{x_1}{w^1_1}, \ldots, \frac{x_m}{w^1_1}, \frac{x_1}{w^1_2}, \ldots, \frac{x_m}{w^1_2}, \ldots, \frac{x_1}{w^1_{L_1}}, \ldots, \frac{x_m}{w^1_{L_1}}\right)\right\} \text{ with } w^1 = \left(w^1_1, \ldots, w^1_{L_1}\right),\ x = (x_1, \ldots, x_m).$$

$$\left\{\frac{w^1}{y} = \left(\frac{w^1_1}{y_1}, \ldots, \frac{w^1_{L_1}}{y_1}, \frac{w^1_1}{y_2}, \ldots, \frac{w^1_{L_1}}{y_2}, \ldots, \frac{w^1_1}{y_s}, \ldots, \frac{w^1_{L_1}}{y_s}\right)\right\} \text{ with } y = (y_1, \ldots, y_s),\ w^1 = \left(w^1_1, \ldots, w^1_{L_1}\right).$$

$$\left\{\frac{\acute{x}}{y} = \left(\frac{\acute{x}_1}{y_1}, \ldots, \frac{\acute{x}_T}{y_1}, \frac{x_1}{y_2}, \ldots, \frac{\acute{x}_T}{y_2}, \ldots, \frac{\acute{x}_1}{y_s}, \ldots, \frac{\acute{x}_T}{y_s}\right)\right\} \text{ with } y = (y_1, \ldots, y_s),\ \acute{x} = (\acute{x}_1, \ldots, \acute{x}_T).$$

$$\left\{\frac{w^2}{y} = \left(\frac{w^2_1}{y_1}, \ldots, \frac{w^2_{L_2}}{y_1}, \frac{w^2_1}{y_2}, \ldots, \frac{w^2_{L_2}}{y_2}, \ldots, \frac{w^2_1}{y_s}, \ldots, \frac{w^2_{L_2}}{y_s}\right)\right\} \text{ with} y = (y_1, \ldots, y_s),\ w^2 = \left(w^2_1, \ldots, w^2_{L_2}\right).$$

$$\left\{\frac{\acute{x}}{w^2} = \left(\frac{\acute{x}_1}{w^2_1}, \ldots, \frac{\acute{x}_T}{w^2_1}, \frac{x_1}{w^2_2}, \ldots, \frac{\acute{x}_T}{w^2_2}, \ldots, \frac{\acute{x}_1}{w^2_{L_2}}, \ldots, \frac{\acute{x}_T}{w^2_{L_2}}\right)\right\} \text{ with } w^2 = \left(w^2_1, \ldots, w^2_{L_2}\right),\ \acute{x} = (\acute{x}_1, \ldots, \acute{x}_T).$$

$$\left\{\frac{x}{z} = \left(\frac{x_1}{z_1}, \ldots, \frac{x_m}{z_1}, \frac{x_1}{z_2}, \ldots, \frac{x_m}{z_2}, \ldots, \frac{x_1}{z_K}, \ldots, \frac{x_m}{z_K}\right)\right\} \text{ with } z = (z_1, \ldots, z_K),\ x = (x_1, \ldots, x_m).$$

$$\left\{\frac{z}{y} = \left(\frac{z_1}{y_1}, \ldots, \frac{z_K}{y_1}, \frac{z_1}{y_2}, \ldots, \frac{z_K}{y_2}, \ldots, \frac{z_1}{y_s}, \ldots, \frac{z_K}{y_s}\right)\right\} \text{ with } y = (y_1, \ldots, y_s),\ z = (z_1, \ldots, z_K).$$

These ratios are defined. A new radial FNDEA-R-I model is presented under constant returns to scale in the input orientation using the input/output ratios for classifying flexible measures to achieve maximum relative efficiency of the DMU in two-stage network.

$$
\begin{array}{ll}
\theta^{*}_{\mathrm{FNDEA-R-I}} = \min \quad \theta, & \\
\text{s.t} \quad \sum_{j=1}^{n} \lambda_j^1 \left(\dfrac{x_{ij}}{y'_{hj}} \right) \le \theta \left(\dfrac{x_{io}}{y'_{ho}} \right), & i = 1,\dots,m, \quad h = 1,\dots,H, \\
\sum_{j=1}^{n} \lambda_j^1 \left(\dfrac{w^1_{l_1 j}}{y'_{hj}} \right) \le \theta \left(\dfrac{w^1_{l_1 O}}{y'_{ho}} \right) + M d^{(1)}_{l_1}, & l_1 = 1,\dots,L_1, \quad h = 1,\dots,H, \\
\sum_{j=1}^{n} \lambda_j^1 d^{(1)}_{l_1} \left(\dfrac{x_{ij}}{w^1_{l_1 j}} \right) \le d^{(1)}_{l_1} \left(\dfrac{x_{iO}}{w^1_{l_1 O}} \right) & i = 1,\dots,m, l_1 = 1,\dots,L_1, \\
\sum_{j=1}^{n} \lambda_j^2 d^{(1)}_{l_1} \left(\dfrac{w^1_{l_1 j}}{y_{rj}} \right) \le d^{(1)}_{l_1} \left(\dfrac{w^1_{l_1 o}}{y_{ro}} \right), & l_1 = 1,\dots,L_1, r = 1,\dots,s, \\
\sum_{j=1}^{n} \lambda_j^2 \left(\dfrac{x'_{tj}}{y_{rj}} \right) \le \theta \left(\dfrac{x'_{to}}{y_{ro}} \right), & t = 1,\dots,T, \ r = 1,\dots,s, \\
\sum_{j=1}^{n} \lambda_j^2 \left(\dfrac{w^2_{l_2 j}}{y_{rj}} \right) \le \theta \left(\dfrac{w^2_{l_2 o}}{y_{ro}} \right) + M d^{(2)}_{l_2}, & l_2 = 1,\dots,L_2, \quad r = 1,\dots,s, \\
\sum_{j=1}^{n} \lambda_j^2 \left(\dfrac{x'_{tj}}{w^2_{l_2 j}} \right) \ge \left(\dfrac{x'_{to}}{w^2_{l_2 o}} \right) - M\left(1 - d^{(2)}_{l_2}\right), & t = 1,\dots,T, \quad l_2 = 1,\dots,L_2 \\
\sum_{j=1}^{n} \lambda_j^1 \left(\dfrac{x_{ij}}{z_{kj}} \right) \le \left(\dfrac{x_{io}}{z_{ko}} \right), & i = 1,\dots,m \ , \ k = 1,\dots,K, \\
\sum_{j=1}^{n} \lambda_j^2 \left(\dfrac{z_{kj}}{y_{rj}} \right) \le \left(\dfrac{z_{ko}}{y_{ro}} \right) & k = 1,\dots,K, \quad r = 1,\dots,s, \\
\sum_{j=1}^{n} \lambda_j^1 = 1, \quad \sum_{j=1}^{n} \lambda_j^2 = 1, & \\
d^{(1)}_{l_1}, d^{(2)}_{l_2} \in \{0,1\}, & l_1 = 1,\dots,L_1, l_2 = 1,\dots,L_2, \\
\lambda_j^1 \ge 0, \ \lambda_j^2 \ge 0, & j = 1,\dots,n,
\end{array}
\tag{4}
$$

In this model, M is a big positive number. Binary variables $d^{(1)}_{l_1}$ and $d^{(2)}_{l_2} \in \{0,1\}$ in the first and second stage are introduced for each l_1 and l_2 to determine that factors l_1 and l_2 are inputs for the first stage and second stage, $d^{(1)}_{l_1} = 0$ and $d^{(2)}_{l_2} = 0$ or outputs for the first stage and second stage $d^{(1)}_{l_1} = 1$ and $d^{(2)}_{l_2} = 1$.

Definition 5. $\mathrm{DMU_o}$ is called a FNDEA-R-I efficient unit if and only if $\theta^{*}_{\mathrm{FNDEA-R-I}} = 1$.

Theorem 1. *Model (4)* is feasible.

Proof: suppose $d^1_{l_1} = 0$, for all l_1, $d^2_{l_2} = 1$, for all l_2 and $\lambda^1_o = 1, \lambda^2_o = 1, \lambda^1_j = 0, \lambda^2_j = 0$; for all j, $j \ne o, \theta = 1$. So, this is a feasible solution for the *Model (4)*.

The technology set $T_{\mathrm{FNDEA-R-I}}$ is defined as follows:

$$
T_{\mathrm{FNDEA-R-I}} = \left\{ \begin{array}{c|c}
\begin{array}{l} (x, w^1, y', z, x', w^2, y) \\ x \succ 0, w^1 \succ 0, y' \succ 0 \\ , z \succ 0, x' \succ 0, w^2 \succ 0, y \succ 0 \end{array} &
\begin{array}{c}
\sum_{j=1}^{n} \lambda_j^1 \left(\dfrac{x_j}{y'_j} \right) \le \left(\dfrac{x}{y'} \right); \left(\begin{array}{c} \sum_{j=1}^{n} \lambda_j^1 \left(\dfrac{w_j^1}{y'_j} \right) \le \left(\dfrac{w^1}{y'} \right) \text{or} \\ \sum_{j=1}^{n} \lambda_j^1 \left(\dfrac{x_j}{w_j^1} \right) \le \left(\dfrac{x}{w^1} \right), \sum_{j=1}^{n} \lambda_j^2 \left(\dfrac{w_j^1}{y_j} \right) \le \left(\dfrac{w^1}{y} \right) \end{array} \right); \\
\sum_{j=1}^{n} \lambda_j^1 \left(\dfrac{x_j}{z_j} \right) \le \left(\dfrac{x}{z} \right); \sum_{j=1}^{n} \lambda_j^2 \left(\dfrac{z_j}{y_j} \right) \le \left(\dfrac{z}{y} \right); \sum_{j=1}^{n} \lambda_j^2 \left(\dfrac{x'_j}{y_j} \right) \le \left(\dfrac{x'}{y} \right); \left(\begin{array}{c} \sum_{j=1}^{n} \lambda_j^2 \left(\dfrac{w_j^2}{y_j} \right) \le \left(\dfrac{w^2}{y} \right) \text{or} \\ \sum_{j=1}^{n} \lambda_j^2 \left(\dfrac{x'_j}{w_j^2} \right) \ge \left(\dfrac{x'}{w^2} \right) \end{array} \right); \lambda_j^1 \ge 0, \lambda_j^2 \ge 0; \text{for all } j
\end{array}
\end{array} \right\}
$$

Theorem 2. *Model (4)* has units-invariant.

Proof: we let,

$$
\begin{aligned}
&x_{ij} \to \alpha_i x_{ij}, \quad \alpha_i \succ 0, \text{for all } i, j,\\
&w^1_{l_1 j} \to \beta_{l_1} w^1_{l_1 j}, \quad \beta_{l_1} \succ 0, \text{for all } l_1, j,\\
&y'_{hj} \to \delta_h y'_{hj}, \quad \delta_h \succ 0, \text{for all } h, j,\\
&x'_{tj} \to \gamma_t x'_{tj}, \quad \gamma_t \succ 0, \text{for all } t, j,\\
&w^2_{l_2 j} \to \eta_{l_2} w^2_{l_2 j}, \quad \eta_{l_2} \succ 0, \text{for all } l_2, j,\\
&y_{rj} \to \sigma_r y_{rj}, \quad \sigma_r \succ 0, \text{for all } r, j,\\
&z_{kj} \to \pi_k z_{kj}, \quad \pi_k \succ 0, \text{for all } k, j.
\end{aligned}
$$

Then we rewrite *Model (4)* as follows:

$$
\begin{aligned}
& & &\theta^*_{FNDEA-R-I} = \min \quad \theta, & &\\
& & &\text{s.t} \quad \sum_{j=1}^{n} \lambda^1_j \left(\frac{\alpha_i x_{ij}}{\delta_h y'_{hj}}\right) \le \theta \left(\frac{\alpha_i x_{io}}{\delta_h y'_{ho}}\right), & & i = 1,\dots,m,\ h = 1,\dots,H,\\
&\text{if } d^{(1)}_{l_1} = 0 \text{ then} & &\sum_{j=1}^{n} \lambda^1_j \left(\frac{\beta_{l_1} w^1_{l_1 j}}{\delta_h y'_{hj}}\right) \le \theta \left(\frac{\beta_{l_1} w^1_{l_1 o}}{\delta_h y'_{ho}}\right), & & l_1 = 1,\dots,L_1,\ h = 1,\dots,H,\\
&\text{if } d^{(1)}_{l_1} = 1 \text{ then} & &\sum_{j=1}^{n} \lambda^1_j \left(\frac{\alpha_i x_{ij}}{\beta_{l_1} w^1_{l_1 j}}\right) \le \left(\frac{\alpha_i x_{io}}{\beta_{l_1} w^1_{l_1 o}}\right), & & i = 1,\dots,m, l_1 = 1,\dots,L_1,\\
& & &\sum_{j=1}^{n} \lambda^2_j \left(\frac{\beta_{l_1} w^1_{l_1 j}}{\sigma_r y_{rj}}\right) \le \left(\frac{\beta_{l_1} w^1_{l_1 o}}{\sigma_r y_{ro}}\right), & & l_1 = 1,\dots,L_1, r = 1,\dots,s,\\
& & &\sum_{j=1}^{n} \lambda^2_j \left(\frac{\gamma_t x'_{tj}}{\sigma_r y_{rj}}\right) \le \theta \left(\frac{\gamma_t x'_{to}}{\sigma_r y_{ro}}\right), & & t = 1,\dots,T,\ r = 1,\dots,s,\\
&\text{if } d^{(2)}_{l_2} = 0 \text{ then} & &\sum_{j=1}^{n} \lambda^2_j \left(\frac{\eta_{l_2} w^2_{l_2 j}}{\sigma_r y_{rj}}\right) \le \theta \left(\frac{\eta_{l_2} w^2_{l_2 o}}{\sigma_r y_{ro}}\right), & & l_2 = 1,\dots,L_2, \quad r = 1,\dots,s,\\
&\text{if } d^{(2)}_{l_2} = 1 \text{ then} & &\sum_{j=1}^{n} \lambda^2_j \left(\frac{\gamma_t x'_{tj}}{\eta_{l_2} w^2_{l_2 j}}\right) \ge \left(\frac{\gamma_t x'_{to}}{\eta_{l_2} w^2_{l_2 o}}\right), & & t = 1,\dots,T, \quad l_2 = 1,\dots,L_2,\\
& & &\sum_{j=1}^{n} \lambda^1_j \left(\frac{\alpha_i x_{ij}}{\pi_k z_{kj}}\right) \le \left(\frac{\alpha_i x_{io}}{\pi_k z_{ko}}\right), & & i = 1,\dots,m\ ,\ k = 1,\dots,K,\\
& & &\sum_{j=1}^{n} \lambda^2_j \left(\frac{\pi_k z_{kj}}{\sigma_r y_{rj}}\right) \le \left(\frac{\pi_k z_{ko}}{\sigma_r y_{ro}}\right), & & k = 1,\dots,K, \quad r = 1,\dots,s,\\
& & &\sum_{j=1}^{n} \lambda^1_j = 1, \quad \sum_{j=1}^{n} \lambda^2_j = 1, & &\\
& & &d^{(1)}_{l_1}, d^{(2)}_{l_2} \in \{0,1\}, \lambda^1_j \ge 0,\ \lambda^2_j \ge 0,\ \text{for all } l_1, l_2\ , j, \quad \theta, \quad \text{free in sign}, & &
\end{aligned}
$$

Thus, we remove $\alpha_i, \beta_{l_1}, \delta_h, \gamma_t, \eta_{l_2}, \sigma_r, \pi_k$ from constraints so *Model (4)* has units-invariant.

3.2.2 | Output-oriented radial FNDEA-R-O model

If the ratio of outputs to inputs is given, we can use FNDEA-R model in the output orientation. To evaluate DMU_o, we present a new radial FNDEA-R-O model under constant returns to scale in the output orientation using the output/ input ratio for classifying flexible measures for maximizing the relative efficiency of the DMU evaluated in two-stage network as follows:

$$
\begin{aligned}
&\varphi^*_{\text{FNDEA-R-O}} = \max \quad \varphi, \\
&\text{s.t} \quad \sum_{j=1}^{n} \lambda_j^1 \left(\frac{y'_{hj}}{x_{ij}} \right) \geq \varphi \left(\frac{y'_{ho}}{x_{io}} \right), && i = 1,\ldots,m, \quad h = 1,\ldots,H, \\
&\sum_{j=1}^{n} \lambda_j^1 \left(\frac{y'_{hj}}{w^1_{l_1 j}} \right) \leq \left(\frac{y'_{ho}}{w^1_{l_1 O}} \right) + M d^{(1)}_{l_1}, && l_1 = 1,\ldots,L_1, \quad h = 1,\ldots,H, \\
&\sum_{j=1}^{n} \lambda_j^1 d^{(1)}_{l_1} \left(\frac{w^1_{l_1 j}}{x_{ij}} \right) \geq d^{(1)}_{l_1} \left(\frac{w^1_{l_1 o}}{x_{io}} \right), && i = 1,\ldots,m, l_1 = 1,\ldots,L_1, \\
&\sum_{j=1}^{n} \lambda_j^2 d^{(1)}_{l_1} \left(\frac{y_{rj}}{w^1_{l_1 j}} \right) \geq d^{(1)}_{l_1} \left(\frac{y_{ro}}{w^1_{l_1 o}} \right), && l_1 = 1,\ldots,L_1, r = 1,\ldots,s, \\
&\sum_{j=1}^{n} \lambda_j^2 \left(\frac{y_{rj}}{x'_{tj}} \right) \geq \varphi \left(\frac{y_{ro}}{x'_{to}} \right), && t = 1,\ldots,T, \ r = 1,\ldots,s, \\
&\sum_{j=1}^{n} \lambda_j^2 \left(\frac{y_{rj}}{w^2_{l_2 j}} \right) \leq \left(\frac{y_{ro}}{w^2_{l_2 o}} \right) + M d^{(2)}_{l_2}, && l_2 = 1,\ldots,L_2, \quad r = 1,\ldots,s, \qquad (5) \\
&\sum_{j=1}^{n} \lambda_j^2 \left(\frac{w^2_{l_2 j}}{x'_{tj}} \right) \geq \varphi \left(\frac{w^2_{l_2 o}}{x'_{to}} \right) - M\left(1 - d^{(2)}_{l_2}\right), && t = 1,\ldots,T, \quad l_2 = 1,\ldots,L_2, \\
&\sum_{j=1}^{n} \lambda_j^1 \left(\frac{z_{kj}}{x_{ij}} \right) \geq \left(\frac{z_{ko}}{x_{io}} \right), && i = 1,\ldots,m, \ k = 1,\ldots,K, \\
&\sum_{j=1}^{n} \lambda_j^2 \left(\frac{y_{rj}}{z_{kj}} \right) \geq \left(\frac{y_{ro}}{z_{ko}} \right), && k = 1,\ldots,K, \quad r = 1,\ldots,s, \\
&\sum_{j=1}^{n} \lambda_j^1 = 1, \quad \sum_{j=1}^{n} \lambda_j^2 = 1, \\
&d^{(1)}_{l_1}, d^{(2)}_{l_2} \in \{0,1\}, && l_1 = 1,\ldots,L_1, l_2 = 1,\ldots,L_2, \\
&\lambda_j^1 \geq 0, \ \lambda_j^2 \geq 0. && j = 1,\ldots,n,
\end{aligned}
$$

Definition 6. DMU_o is called a FNDEA-R-O efficient unit if and only if $\varphi^*_{\text{FNDEA-R-O}} = 1$. The efficiency corresponding to DMU_o is defined as $\frac{1}{\varphi^*_{\text{FNDEA-R-O}}}$.

The technology set $T_{\text{FNDEA-R-O}}$ is defined as follows:

$$
T_{\text{FNDEA-R-O}} = \left\{ \begin{array}{c|c} \begin{array}{c} (x, w^1, y', z, x', w^2, y) \\ x \succ 0, w^1 \succ 0, y' \succ 0, z \succ 0, \\ x' \succ 0, w^2 \succ 0, y \succ 0 \end{array} & \begin{array}{c} \sum_{j=1}^{n} \lambda_j^1 \left(\frac{y'_j}{x_j}\right) \geq \left(\frac{y'}{x}\right); \left(\begin{array}{c} \sum_{j=1}^{n} \lambda_j^1 \left(\frac{y'_j}{w^1_j}\right) \leq \left(\frac{y'}{w^1}\right) \text{or} \\ \sum_{j=1}^{n} \lambda_j^1 \left(\frac{w^1_j}{x_j}\right) \geq \left(\frac{w^1}{x}\right), \sum_{j=1}^{n} \lambda_j^2 \left(\frac{y_j}{w^1_j}\right) \geq \left(\frac{y}{w^1}\right) \end{array} \right), \\ \sum_{j=1}^{n} \lambda_j^1 \left(\frac{z_j}{x_j}\right) \geq \left(\frac{z}{x}\right); \sum_{j=1}^{n} \lambda_j^2 \left(\frac{y_j}{z_j}\right) \geq \left(\frac{y}{z}\right); \sum_{j=1}^{n} \lambda_j^2 \left(\frac{y_j}{x'_j}\right) \geq \left(\frac{y}{x'}\right); \left(\begin{array}{c} \sum_{j=1}^{n} \lambda_j^2 \left(\frac{y_j}{w^2_j}\right) \leq \left(\frac{y}{w^2}\right) \text{or} \\ \sum_{j=1}^{n} \lambda_j^2 \left(\frac{w^2_j}{x'_j}\right) \geq \left(\frac{w^2}{x'}\right) \end{array} \right), \lambda_j^1 \geq 0, \lambda_j^2 \geq 0; \text{for all } j \end{array} \end{array} \right\}
$$

Theorem 3. *Model (5)* is feasible.

Proof: suppose $d^1_{l_1} = 0$, for all l_1, $d^2_{l_2} = 1$, for all l_2 and $\lambda_o^1 = 1, \lambda_o^2 = 1, \lambda_j^1 = 0, \lambda_j^2 = 0$; for all j, $j \neq o, \varphi = 1$. So, this is a feasible solution for the *Model (5)*.

Definition 7. *Model (5)* has units-invariant.

4 | Numerical Example

In this section, we use models of input and output orientation and solve the models introduced here in two-stage network for flexible measures classification. Parameters are random data are listed in *Table 1* obtained column-wise using GAMS software by uniform distribution such as uniform (a, b) where a, b are the upper and lower bounds of the uniform distribution. The intervals are given at the last row of Table 1. In total, we have 30 DMUs, three inputs to the first stage (X1, X2, X3), two outputs to the first stage ($\acute{Y}1$, $\acute{Y}2$), leaving the system, two intermediate measures (Z1, Z2), two final outputs from the second stage (Y1, Y2), the second stage has its own inputs ($\acute{X}1$, $\acute{X}2$), one flexible measure to the first stage w^1 and one flexible measure to the second stage w^2. In the presented DEA-R models, all inputs and outputs are positive and the ratios of inputs to outputs and vice versa are defined.

Table 1. Data with flexible measures in general two-stage network.

DMU	X1	X2	X3	Z1	Z2	$\dot{Y}1$	$\dot{Y}2$	$\dot{X}1$	$\dot{X}2$	Y1	Y2	w^1	w^2
DMU01	22.3	13.2	54.6	110.1	66.1	21.8	44.6	18	31	13.3	12.5	22.02	97.73
DMU02	68.3	83.3	15.8	75.4	116.4	19.8	12	19.6	25.8	2.4	18.2	69.03	45.17
DMU03	52	19.2	31.2	94.3	59.9	47.3	47.4	11.5	22.5	23.3	36	48.53	21.21
DMU04	31.8	52	40.3	66.4	127.2	40.5	35.8	16.8	37.1	30	19.5	31.08	71.07
DMU05	95.3	42	29	108.9	52.3	75	22.5	14	27.2	15.8	16.7	30.45	98.56
DMU06	52.8	6.1	62.6	102.4	78.8	69.6	27	14.8	44.9	12.6	20.4	25.68	30.01
DMU07	50.5	9.3	48.7	124.6	120.6	52.2	49.8	5.9	38.5	9.5	20.5	94.49	51.58
DMU08	80.1	17.4	58.4	64.5	131.2	37.7	14.6	10.3	65.6	16.7	39.9	69.94	60.59
DMU09	53.9	14	36.9	129.8	122.1	60.9	24.1	11.9	49.5	16.8	55	14.70	53.98
DMU10	20.9	90.5	48.8	66.4	132.5	92.2	68.7	10.1	54.5	10	28.3	45.01	42.47
DMU11	82.5	71.1	16.8	71.9	138.9	47.7	60.7	5.6	19.1	19.7	33.6	79.87	38.93
DMU12	27	10.6	25.6	51.9	84.4	47.3	63.3	11	39.6	12.2	43.7	50.51	98.24
DMU13	49.6	10.7	20.6	125.5	97.3	75.3	32.6	17.7	38.9	18.9	44.7	79.38	31.99
DMU14	55.7	19.4	46.6	91.5	117.3	79	60.3	11.8	26.4	7.5	38.7	63.36	63.26
DMU15	55.1	18.2	52.5	90.1	61	72.2	24.9	17	33.5	17.2	43.9	79.15	33.70
DMU16	66.3	8	34.9	131.1	63.7	57	30.7	10.7	52.5	11.2	15.5	54.78	82.75
DMU17	93.3	6.3	43.5	53.5	133.9	38.6	32.1	13.4	45	19.7	15.4	71.17	55.20
DMU18	10.8	11.9	31.5	118.7	89.4	34.9	23.6	11	67.3	82	20.5	27.51	49.25
DMU19	98.5	6.8	21.3	75.4	133	28	28.9	16.1	26.7	9.5	20.3	86.83	63.90
DMU20	27.8	17.1	24.9	81	52.2	30.6	14.3	16.9	35.3	17.4	15.4	40.47	30.85
DMU21	42	7.2	59.7	98.4	147.5	29.2	39.4	14.8	42.3	10.7	44.5	35.18	56.28
DMU22	98.7	88.5	51	132.8	60.6	57.3	69.3	19.8	61.9	19.9	33.3	34.60	62.91
DMU23	53.5	15.6	25.7	93.5	121.6	31.3	34.6	19.7	56.5	13	47.6	59.20	53.92
DMU24	25.1	16.7	56.8	81.6	145.6	62.1	74.8	11.7	17.4	7.6	29.9	20.51	29.87
DMU25	96.3	15.3	45.1	120.5	133.6	25.7	56.8	19.7	16.9	14.9	38.5	51.24	96.05
DMU26	97.9	6.8	53.1	103.8	89.8	45.7	49.6	17.7	56.3	4.9	12.5	68.16	17.18
DMU27	37.4	15	15.5	63.1	128.2	53.1	22	5.5	57.7	5.8	11.8	66.16	54.89
DMU28	70	12.8	21.5	126.1	97.2	28.3	44.3	11.4	56.7	4.9	47.2	56.60	48.17
DMU29	24	55.8	33.8	91.2	82.6	73.7	76.2	19.4	42.4	7.8	10.3	64.31	62.21
DMU30	48.6	18.6	55.9	126	73.8	15.4	57.6	17.1	76.2	13.2	25.6	31.26	65.47
	[10,100]	[5,95]	[15,65]	[50,140]	[50,150]	[15,95]	[10,80]	[5,20]	[15,80]	[2,85]	[10,60]	[14,95]	[15,100]

The results of the FNDEA *Model (3)*, FNDEA-R-I *Model (4)*, FNDEA-R-O *Model (5)*, in general two-stage to classify flexible measures are listed in *Table 2*. In FNDEA-R-I model most DMUs choose the flexible measure w^1 in the first stage as an input measure ($d_{l_1}^{*(1)} = 0$) but, in FNDEA model and FNDEA-R-O model most DMUs select the flexible measure w^1 in the first stage as an output measure ($d_{l_1}^{*(1)} = 1$). In FNDEA-R-I model most DMUs select the flexible measure w^2 in the second stage as an input measure ($d_{l_2}^{*(2)} = 0$) but, in FNDEA model and FNDEA-R-O model most DMUs determine the flexible measure w^2 in the second stage as an output measure ($d_{l_2}^{*(2)} = 1$).

Table 2. The results of the FNDEA Model (3), FNDEA-R-I Model (4) and FNDEA-R-O Model (5) in two-stage network DEA-R.

DMU	θ^*_{FNDEA}	$d^{*(1)}$	$d^{*(2)}$	$\theta^*_{DEA-R-I}$	$d^{*(1)}$	$d^{*(2)}$	$\frac{1}{\varphi^*_{DEA-R-O}}$	$d^{*(1)}$	$d^{*(2)}$
1	0.8614	1	0	1	0	0	0.862	1	1
2	0.9851	0	1	1	0	0	1	1	1
3	1	0	1	1	1	0	1	1	1
4	0.7385	1	0	1	0	0	0.801	1	1
5	0.7473	1	0	0.797	1	0	1	1	1
6	1	1	0	1	0	0	1	1	1
7	1	1	0	1	0	0	1	1	1
8	0.7238	1	0	1	1	0	0.755	1	1
9	1	1	0	1	0	0	1	1	1
10	1	1	1	1	1	0	1	1	1
11	1	1	1	1	0	0	1	1	1
12	1	1	1	1	1	0	1	1	1
13	1	1	1	1	0	0	1	1	1
14	1	1	0	0.899	1	1	0.818	1	1
15	0.8966	1	1	1	0	1	1	1	1
16	0.9918	1	0	1	0	1	1	1	0
17	1	1	0	1	1	1	1	1	1
18	1	1	0	1	0	1	1	1	1
19	1	1	0	1	0	0	1	1	0
20	0.9129	1	1	0.961	1	0	0.444	1	1
21	1	1	1	1	1	0	1	1	1
22	0.7436	1	1	1	0	1	0.68	1	1
23	0.7739	1	1	0.993	0	0	0.552	1	1
24	1	1	1	1	0	0	1	1	1
25	1	1	1	1	0	0	1	1	1
26	1	1	1	1	1	0	1	1	1
27	1	1	1	1	0	0	1	1	0
28	0.984	1	1	1	0	0	1	1	0
29	1	1	1	1	0	0	1	1	1
30	0.6173	1	1	0.772	1	0	0.558	1	0

One flexible measure status in the input-oriented and output-oriented FNDEA-R models in the two-stage general network structure may have different conclusions and it is expectable that the flexible measure is chosen as an input measure in one model and an output measure in another model. In the input-oriented FNDEA-R-I model most DMUs choose the flexible measure in the first and second stage as input measures and in the output-oriented FNDEA-R-O model most DMUs choose the flexible measure in the first and second stage as output measures. In these models each flexible measure be an input for some DMUs and output for others to achieve a maximum relative efficiency of the DMU. It is noticeable that because of the capability of FNDEA-R models to overcome underestimation of efficiency and pseudo-inefficiency, the efficiency scores obtained from the FNDEA-R-I model are greater than or equal to the efficiency scores obtained from the FNDEA model, indicating that FNDEA-R-I model versus FNDEA model prevent efficiency underestimation and pseudo inefficiency issues.

5 | Conclusion

In conventional DEA models, every variable must be determined as an output or input prior to efficiency measurement. Still, in practice, some measure of which the status is flexible and the data are the input/output ratio data or the vice versa, which is critical for decision makers in two-stage network. Novel radial FNDEA-R models are proposed in the presence of flexible measures for the third category of ratio data using the input/output ratio and vice versa in the input and output orientation with constant returns to scale in general two-stage network DEA-R to classify flexible measures where each flexible measure has an input or output role to achieve the highest relative efficiency of DMU under evaluation. The models are good with key disadvantages in DEA models such as inaccurate estimates of efficiency, weak efficiency and pseudo-inefficiency. The efficiency scores obtained from the FNDEA-R models are greater than or equal to the efficiency scores obtained from the FNDEA model. The FNDEA-R models provide a bigger feasible space for selecting the corresponding weight of output/input ratios and input/output ratios. Therefore, all the input-to-output ratios or vice versa are included to measure the DMU efficiency. The condition of the flexible measure in the input-oriented and output-oriented FNDEA-R models in the general two-stage network structure may have different conclusions and it is expectable to select a flexible measure as an input measure for one model while an output for another. In proposed models in DEA-R flexible measures are determined as input or output in the numerator or denominator of ratios to increase relative efficiency. By solving a model based on the radial model in DEA and DEA-R, both flexible measures are classified and the relative efficiency of the units is measured. In the radial models in DEA-R, all inputs and outputs should be positive and the ratios of inputs to outputs and vice versa should be defined. In the proposed radial models in DEA and DEA-R, constraints are linear, although there are binary variables in these models. The proposed radial models in DEA and DEA-R are always feasible and bounded. The proposed radial models in DEA and DEA-R have units-invariant. We can develop the FNDEA-R models for situations that imprecise factors and nonpositive data are present.

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Data Availability

The numerical data and model parameters used in this study are either generated within the scope of the proposed models or embedded in the illustrative example section. No additional datasets were generated or analyzed.

Conflicts of Interest

The author declares that there are no known competing financial or non-financial interests that could have influenced the outcomes or interpretations in this research.

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