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A Theoretical and Computational Study of a Defuzzification Function for the Generalized Fuzzy Single-Depot Multiple Traveling Salesman Problem

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Abstract

This study addresses the single-depot multiple Traveling Salesman Problem (mTSP) under uncertainty, where travel costs are expressed as generalized trapezoidal fuzzy numbers. A two-stage parametric approach is adopted: first, a defuzzification function M_α converts the fuzzy costs into crisp values; second, a corrected Mixed-Integer Linear Programming (MILP) model, equipped with strict sequencing constraints to enforce tour-length limits, is solved to obtain optimal routes. The primary theoretical contribution is a rigorous analysis of the defuzzification function M_α, including proofs of its continuity, strict monotonic decrease with respect to the risk parameter α, positivity under mild conditions, convexity, and concavity properties, thereby establishing a sound mathematical basis for its use. The method is demonstrated on a 7-city, 2-salesman instance with tour-length bounds $k = 2$ and $w = 6$. Experiments for $\alpha = 0.0, 0.1, 0.2, 0.3, 0.4, 0.5$ reveal that total cost declines strictly as α increases (from 116.62 to 8.81), and the optimal tour configuration switches at $\alpha = 0.3$, illustrating how the parameter α acts as a risk-attitude lever that can fundamentally alter operational plans. The approach provides decision-makers with a transparent, quantitative tool to explore cost–risk trade-offs in fuzzy routing problems.

Keywords: Multiple traveling salesman problem, Generalized trapezoidal fuzzy numbers, Defuzzification function, Mixed-integer linear programming, Risk analysis.

1 | Introduction

The multiple Traveling Salesman Problem (mTSP) generalizes the classical Traveling Salesman Problem (TSP) by deploying m salesmen, all starting from a common depot, to visit a set of n cities such that each city is visited exactly once by a single salesman, and all salesmen eventually return to the depot. The objective is to minimize the total cost of all tours [1], [2]. Owing to its wide-ranging applications in logistics, vehicle routing

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[3], flight scheduling [4], hot rolling scheduling in the steel industry [5], drone delivery systems [6], and energy collection network design [7], the mTSP has consistently attracted significant research attention.

Solution methods for the mTSP fall broadly into two categories: Exact algorithms and approximate algorithms. Exact approaches, including Mixed-Integer Linear Programming (MILP) and branch-and-bound [8], [9], guarantee optimality, yet their computational burden escalates dramatically with problem size [10], [11]. By contrast, approximate methods, heuristics, and metaheuristics can produce high-quality solutions within reasonable time. Genetic Algorithms (GA) [12–14], Ant Colony Optimization (ACO) [15–17], memetic algorithms [18], hybrid methods [19], tabu search [20], and heart-inspired algorithms [21] are among the metaheuristics that have been successfully applied to the mTSP and related problems. A comprehensive overview of mTSP formulations and solution procedures is provided by Bektas [2].

Despite these advances, most classical mTSP models treat travel costs as deterministic. In reality, however, factors such as traffic congestion, weather conditions, fuel price fluctuations, and vehicle type introduce significant uncertainty into cost parameters. Fuzzy set theory offers a powerful framework for modelling such uncertainty [22], [23]. Within the realm of fuzzy TSP and mTSP, several studies have employed ordinary triangular or trapezoidal fuzzy numbers (with a fixed height of 1). For instance, Feng and Liao [24] applied a hybrid evolutionary fuzzy learning scheme to the TSP; Trigui et al. [25] used fuzzy logic to address a multi-objective, multi-depot mTSP; Shi et al. [26] solved a home health care routing problem with fuzzy demand. Nevertheless, the use of generalized trapezoidal fuzzy numbers, which permit a flexible membership height ($\mu \leq 1$), has received comparatively less attention. This class of fuzzy numbers provides greater flexibility for modelling uncertainties whose degree of confidence may vary.

A critical step in solving fuzzy optimization problems is the conversion of fuzzy numbers into crisp values via a defuzzification (or ranking) function. A valid defuzzification function should satisfy certain logical properties such as continuity, monotonicity, and positivity; however, the theoretical underpinnings of these functions are often not rigorously examined in the literature. In the present work, we employ a parametric defuzzification function M_α introduced by Farnam and Darehmiraki [27] to handle the mTSP under a generalized trapezoidal fuzzy environment. The primary contribution of this study is the theoretical development of this function: We prove theorems concerning its continuity and differentiability (*Theorem 1*), strict monotonic decrease (*Theorem 2*), boundedness and positivity under sufficient conditions (*Theorem 3*), and convexity/concavity properties (*Theorem 4*). In doing so, we establish a firm mathematical foundation for using M_α as a reliable tool in fuzzy decision-making.

The second contribution is the deployment of this function within a two-stage algorithm for solving the fuzzy mTSP, together with a sensitivity analysis of the parameter α as a risk-control lever. In the first stage, the fuzzy cost matrix is transformed into a crisp matrix via M_α; in the second stage, a standard MILP model (with subtour elimination and tour-length constraints) is solved to determine optimal routes. The parameter α represents the fuzzy cut level and mirrors the decision-maker's attitude, ranging from fully conservative $\alpha = 0$ to highly optimistic α close to μ. Numerical experiments on a 7-city instance with two salesmen illustrate how varying α not only yields a monotonic reduction in total cost but can also alter the structure of the optimal tours. Thus, the approach provides managers with a quantitative tool for evaluating different risk scenarios.

The remainder of the paper is organized as follows. Section 2 presents the problem definition, notation, and the MILP formulation. Section 3 is devoted to the theoretical development and proofs of the properties of the defuzzification function M_α. Section 4 describes the numerical example in detail and analyzes the results. Finally, Section 5 concludes the paper and outlines directions for future research.

2 | Problem Definition and Notation

The single-depot mTSP considered in this study is defined on a complete directed graph $G = (V, A)$, where $V = \{1, 2, \dots, n\}$ is the set of nodes (cities) and $A = \{(i, j) \mid i, j \in V, i \neq j\}$ is the set of arcs. Node 1 is designated as the depot, at which m salesmen are initially located. The remaining nodes, indexed 2 through n, represent non-depot cities that must be served. Each salesman must depart from the depot, visit a sequence of non-depot cities, and ultimately return to the depot. Every non-depot city must be visited exactly once by exactly one salesman. The objective is to determine a set of m tours that minimizes the total travel cost.

In real-world settings, travel costs are seldom known with certainty. To capture the imprecision arising from factors such as traffic conditions, weather, and fuel price volatility, we model the cost of traversing each arc (i, j) as a generalized trapezoidal fuzzy number:

$$\tilde{c}_{ij} = \langle (a_{ij}, b_{ij}, c_{ij}, d_{ij}); \mu_{ij} \rangle,$$

where $0 < a_{ij} \leq b_{ij} \leq c_{ij} \leq d_{ij}$ are the parameters of the trapezoid, and $\mu_{ij} \in (0,1]$ is the height (maximum membership degree). Asymmetric costs are permitted, i.e., $\tilde{c}_{ij}$ and $\tilde{c}_{ji}$ may differ.

A two-stage solution approach is adopted. In the first stage, a parametric defuzzification function M_α (defined in Section 3 is applied to each fuzzy cost $\tilde{c}_{ij}$ to obtain a crisp cost $c_{ij} = M_\alpha(\tilde{c}_{ij})$. The resulting deterministic cost matrix $[c_{ij}]_{n\times n}$ is then fed into a MILP formulation, which is solved in the second stage. The MILP model, referred to as the corrected formulation, is presented below.

Parameters:

n: total number of cities (city 1 is the depot).

m: number of salesmen.

k: minimum number of non-depot cities that a single salesman must visit.

w: maximum number of non-depot cities a salesman may visit $(k \leq w)$.

Decision variables:

$x_{ij} \in \{0,1\}$ for $(i, j) \in A$: Equals one if the (i, j) arc is traversed, and zero otherwise.

$u_i \in Z^+ \text{for} (i = 2, \dots, n)$: Auxiliary variable indicating the sequential position of city i within its tour. For the depot, $u_1 = 1$ is fixed.

Model (MILP – after defuzzification):

$$\text{Minimize } Z = \sum_{i=1}^{n} \sum_{j=1 j\neq i}^{n} c_{ij} x_{ij}. \tag{1}$$

s. t.

$$\sum_{j=2}^{n} x_{1j} = m, \quad \sum_{j=2}^{n} x_{j1} = m. \tag{2}$$

$$\sum_{i=1, i\neq j}^{n} x_{ij} = 1, \quad j = 2, \dots, n. \tag{3}$$

$$\sum_{j=1, j\neq i}^{n} x_{ij} = 1, \quad i = 2, \dots, n. \tag{4}$$

$$x_{ii} = 0, \quad i = 1, \dots, n. \tag{5}$$

$$u_i - u_j + w x_{ij} \leq w - 1, i, j = 2, \dots, n; \; i \neq j. \tag{6}$$

$$u_j - u_i \geq w x_{ij} - (w - 1), i, j = 2, \dots, n; \; i \neq j. \tag{7}$$

$$u_j - u_i \leq w + 1 - wx_{ij}, i, j = 2, \dots, n;\ i \neq j. \tag{8}$$

$$u_i + (w-2)x_{1i} - x_{i1} \leq w - 1, i = 2, \dots, n. \tag{9}$$

$$u_i + x_{1i} + (2-k)x_{i1} \geq 2, i = 2, \dots, n. \tag{10}$$

$$1 \leq u_i \leq w, \quad i = 2, \dots, n. \tag{11}$$

$$u_1 = 1. \tag{12}$$

$$x_{ij} \in \{0,1\}, \quad (i,j) \in A. \tag{13}$$

Explanation of constraints:

I. *Constraint (2)* ensures exactly m salesmen leave from and return to the depot.

II. *Constraints (3)* and *(4)* enforce flow conservation: each non-depot city is entered and left exactly once.

III. *Constraint (5)* prohibits self-loops.

IV. *Constraint (6)* is the standard Miller–Tucker–Zemlin (MTZ) inequality, which prevents subtours among non-depot cities by enforcing a strict ordering if $x_{ij} = 1$.

V. *Constraints (7)* and *(8)* together form a linearization of the equality $u_j = u_i + 1$ whenever $x_{ij} = 1$. These constraints ensure that the u-variables advance in unit steps along each tour, thereby making the u-value of the last city before returning to the depot equal to the exact number of non-depot cities in that tour. This strict sequencing is essential for the correct enforcement of the minimum tour length k.

VI. *Constraint (9)* forces $u_i = 1$ if city i is the first non-depot city visited after the depot $x_{1i} = 1$.

VII. *Constraint (10)* forces $u_i \geq k$ if city i is the last non-depot city visited before returning to the depot $x_{i1} = 1$, and together with *Constraints (7)* and *(8)* guarantees that the tour contains at least k cities.

VIII. *Constraints (11)* and *(12)* bound the auxiliary variables.

IX. *Constraint (13)* defines the binary nature of the routing variables.

For the numerical experiment presented in this paper, we set $n = 7, m = 2, k = 2$, and $w = 6$. These values are chosen to ensure feasibility given the six non-depot cities, while still allowing both balanced and unbalanced allocations of cities to salesmen. The crisp cost matrix c_{ij} is obtained by applying the defuzzification function M_α (Section 3) to the fuzzy costs listed in Section 3. The resulting MILP is solved using the "intlinprog" solver in MATLAB R2018a.

3 | Theoretical Development of the Defuzzification Function M_α

In this section, we establish the theoretical foundation of the parametric defuzzification function M_α that is employed in the first stage of the proposed solution algorithm. Building upon the work of Farnam and Darehmiraki [27], we prove several essential properties that qualify M_α as a valid ranking function for generalized trapezoidal fuzzy numbers. These properties, continuity, strict monotonicity, boundedness, and positivity, are crucial to guarantee that the subsequent optimization phase yields meaningful and risk-adjustable crisp costs.

3.1 | Preliminaries: Generalized Trapezoidal Fuzzy Numbers

A generalized trapezoidal fuzzy number $\widetilde{A}$ is characterized by a quadruple (a, b, c, d) with $0 < a \leq b \leq c \leq d$ and a height $\mu \in (0,1]$. It is denoted as $\widetilde{A} = \langle (a, b, c, d); \mu \rangle$. Its membership function is

$$\mu_{\tilde{A}}(x) = \begin{cases} \mu.\dfrac{x-a}{b-a} & a \le x < b, \\ \mu & b \le x \le c, \\ \mu.\dfrac{d-x}{d-c} & c < x \le d, \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

In the mTSP under study, the cost of traversing any arc (i,j) is a positive generalized trapezoidal fuzzy number $\tilde{c}_{ij} = \langle (a_{ij}, b_{ij}, c_{ij}, d_{ij}); \mu_{ij} \rangle$.

Definition 1 (parametric defuzzification function M_α ([27])). For a generalized trapezoidal fuzzy number $\tilde{A} = \langle (a,b,c,d); \mu \rangle$ and a parameter $\alpha \in [0,\mu]$, the function $M_\alpha(\tilde{A})$ is defined as:

$$M_\alpha(\tilde{A}) = (a+d)(\mu-\alpha) + (b+c-a-d)\,\frac{\mu^2-\alpha^2}{2\mu}. \tag{15}$$

The parameter α acts as a fuzzy cut level. By increasing α, the contribution of the core region of the fuzzy number (where membership is high) is emphasised, while the influence of the boundaries (where membership is low) diminishes.

3.2 | Characterization Theorems

Theorem 1 (continuity and differentiability). As a function of α, $M_\alpha(\tilde{A})$ is continuous on $[0,\mu]$ and infinitely differentiable on $(0,\mu)$.

Proof: Expanding *Eq. (15)* yields

$$M_\alpha(\tilde{A}) = (a+d)\mu - (a+d)\alpha + \frac{b+c-a-d}{2\mu}(\mu^2-\alpha^2)$$

$$= (a+d)\mu + \frac{b+c-a-d}{2}\mu - (a+d)\alpha - \frac{b+c-a-d}{2\mu}\alpha^2.$$

This expression is a second-degree polynomial in α. Since polynomials are continuous and possess derivatives of all orders on R, the claim follows. The first and second derivatives are

$$\frac{dM_\alpha}{d\alpha} = -(a+d) - \frac{b+c-a-d}{\mu}\alpha,$$

$$\frac{d^2M_\alpha}{d\alpha^2} = -\frac{b+c-a-d}{\mu}.$$

Theorem 2 (strict monotonicity). For any positive generalized trapezoidal fuzzy number $\tilde{A}$ $(a>0)$, the function $M_\alpha(\tilde{A})$ is strictly decreasing with respect to α on $[0,\mu]$.

Proof: We show that $\frac{dM_\alpha}{d\alpha} < 0$ for all $\alpha \in (0,\mu)$.

$$\frac{dM_\alpha}{d\alpha} = -(a+d) - \frac{b+c-a-d}{\mu}\alpha.$$

This derivative is an affine function of α. Its value at the endpoints of the interval is

$$\frac{dM_\alpha}{d\alpha}\Big|_{\alpha=0} = -(a+d) < 0,$$

$$\frac{dM_\alpha}{d\alpha}\Big|_{\alpha=\mu} = -(a+d) - (b+c-a-d) = -(b+c) < 0.$$

(Since $a + d > 0$ and $b + c > 0$), both endpoint values are strictly negative. Because the derivative is linear in α, its sign on the whole interval $[0, \mu]$ is determined by its values at the endpoints and its slope.

I. If $b + c - a - d \geq 0$, the slope $-\frac{b+c-a-d}{\mu} \leq 0$; thus, the derivative is a non-increasing function of α. Its maximum on $[0, \mu]$ occurs at $\alpha = 0$, which is $-(a + d) < 0$. Hence the derivative remains negative.

II. If $b + c - a - d < 0$, the slope is positive, so the derivative is increasing. Its maximum on $[0, \mu]$ occurs at $\alpha = \mu$, which equals $-(b + c) < 0$. Therefore, the derivative is negative throughout.

In both cases, $\frac{dM_\alpha}{d\alpha} < 0$ for all $\alpha \in (0, \mu)$. Consequently, M_α is strictly decreasing on $[0, \mu]$.

Theorem 3 (positivity and boundedness). Let $\widetilde{A}$ be a positive generalized trapezoidal fuzzy number. Then

$$0 = M_\mu(\widetilde{A}) < M_\alpha(\widetilde{A}) \leq M.(\widetilde{A}) \text{ for all } \alpha \in [0, \mu).$$

Moreover, $M_\alpha(\widetilde{A}) > 0$ holds for every $\alpha < \mu$ without any additional condition.

Proof: The strictly decreasing property from *Theorem 2* immediately gives $M_\mu(\widetilde{A}) < M_\alpha(\widetilde{A}) < M.(\widetilde{A})$ for $\alpha \in (0, \mu)$. Substituting $\alpha = \mu$ into *Factor (15)* yields $M_\mu(\widetilde{A}) = 0$. To prove that $M_\alpha(\widetilde{A}) > 0$ for $\alpha < \mu$, *Factor (15)* as

$$M_\alpha(\widetilde{A}) = (\mu - \alpha)\left[a + d + (b + c - a - d)\frac{\mu + \alpha}{2\mu}\right].$$

The factor $\mu - \alpha$ is strictly positive for $\alpha < \mu$. Let $B(\alpha)$ denote the bracketed expression. $B(\alpha)$ is a linear function of α with slope $\frac{b+c-a-d}{2\mu}$.

I. If $b + c - a - d \geq 0$, B is non-decreasing. Its minimum on $[0, \mu]$ occurs at $\alpha = 0$:

$$B(0) = a + d + \frac{b + c - a - d}{2} = \frac{a + d}{2} + \frac{b + c}{2} > 0.$$

II. If $b + c - a - d < 0$, B is decreasing. Its minimum occurs at $\alpha = \mu$:

$$a + d + (b + c - a - d).1 = b + c > 0.$$

In both cases, $B(\alpha) > 0$ for every $\alpha \in [0, \mu]$. Hence $M_\alpha(\widetilde{A}) = (\mu - \alpha)B(\alpha) > 0$ for all $\alpha < \mu$. The upper bound is $M.(\widetilde{A}) = (a + d)\mu + \frac{b+c-a-d}{2}\mu$.

It should be noted that $M_\mu(\widetilde{A}) = 0$ is a limiting value that is not attainable in practice, because the parameter α must be chosen strictly less than $\min_{ij} \mu_{ij}$ to prevent any crisp cost from becoming zero or negative.

Theorem 4. The function $M_\alpha(\widetilde{A})$ is

I. Strictly concave on $[0, \mu]$ if $b + c > a + d$,

II. Linear if $b + c = a + d$,

III. Strictly convex if $b + c < a + d$.

Proof: The second derivative is $\frac{d^2M_\alpha}{d\alpha^2} = -\frac{b+c-a-d}{\mu}$. The sign of this constant determines the curvature:

I. If $b + c - a - d > 0$, the second derivative is negative, implying strict concavity.

II. If $b + c - a - d = 0$, the second derivative is zero, so the function is linear.

III. If $b + c - a - d < 0$, the second derivative is positive, giving strict convexity.

3.3 | Verification on Sample Data

To illustrate the theoretical properties, we evaluate M_α for three arcs taken from the numerical example (Section 4). The results are summarized in *Table 1*.

Table 1. Defuzzified costs for sample arcs at different α levels.

α	$(\tilde{c}_{14})$ **linear**: $(a+d=b+c)$	$(\tilde{c}_{15})$ **convex**: $(a+d>b+c)$	$(\tilde{c}_{12})$ **concave**: $(a+d<b+c)$
0.0	13.5200	7.3500	8.9600
0.1	11.8300	5.8640	7.6971
0.2	10.1400	4.3860	6.4286
0.3	8.4500	2.9160	5.1543
0.4	6.7600	1.4540	3.8743
0.5	5.0700	0.0000	2.5886

For $\tilde{c}_{14} = \langle(5.7,7.3,9.6,11.2); 0.8\rangle$, we have $a+d = 16.9 = b+c$, hence the function is linear. For $\tilde{c}_{15} = \langle(4.2,6.1,8.4,10.7); 0.5\rangle$, $a+d = 14.9 > 14.5 = b+c$, so the function is convex. For $\tilde{c}_{12} = \langle(3.5,5.2,7.8,9.1); 0.7\rangle$, $a+d = 12.6 < 13.0 = b+c$, making the function concave. In all cases, monotonic decrease with α is evident. Positivity is maintained for $\alpha < \mu$; the zero value at $\alpha = 0.5$ for $\tilde{c}_{15}$ is consistent with $\alpha = \mu_{15}$, confirming that the cut level must be kept below the minimum height across all arcs.

3.4 | Interpretation and Implications

The established properties endow the defuzzification function M_α with a clear decision-theoretic interpretation. The parameter α controls the degree of optimism/pessimism embedded in the crisp equivalent cost:

I. At $\alpha = 0$, the entire support of the fuzzy number contributes to the crisp value, yielding the most conservative (i.e., highest) cost estimate.

II. As α increases, the influence of low-membership regions diminishes, and the crisp cost decreases monotonically.

III. As $\alpha \to \mu$, the crisp cost approaches zero, which represents an extreme, unattainable optimistic scenario.

Consequently, α serves as a continuous risk-adjustment lever: A risk-averse decision-maker would choose a small α, while a risk-seeking decision-maker might select a larger α. The convexity/concavity of the function also indicates that the rate of cost reduction is not constant but depends on the asymmetry of the trapezoidal number.

These theoretical guarantees ensure that the defuzzification stage does not introduce artifacts (e.g., negative costs) that could compromise the MILP solution. In the following section, the complete two-stage algorithm is applied to a 7-city mTSP instance, and the impact of α on both the cost and the structure of optimal tours is thoroughly analyzed.

4 | Numerical Experiment and Analysis

In this section, the two-stage algorithm is applied to a concrete instance of the fuzzy mTSP. We first detail the problem data, then illustrate the defuzzification process, present the optimal solutions obtained for six distinct values of the risk parameter α, and conclude with a discussion of the managerial insights offered by the approach. All computations were performed in MATLAB R2018a on a standard desktop computer (Intel Core i7, 16 GB RAM) using the `intlinprog` solver. The complete code is available from the authors upon request.

4.1 | Problem Instance

The instance consists of n=7 cities, with city 1 serving as the common depot. There are m=2 salesmen. The minimum and maximum number of non-depot cities that may be assigned to a single salesman are set to k=2 and w=6, respectively. With only six non-depot cities $(2, \ldots, 7)$, these bounds ensure feasibility while still permitting imbalanced allocations (e.g., a tour of length 4 together with a tour of length 2, or the symmetric 3 + 3 split).

The travel costs are modelled as generalized trapezoidal fuzzy numbers $\tilde{c}_{ij} = \langle (a_{ij}, b_{ij}, c_{ij}, d_{ij}); \mu_{ij} \rangle$ with $a_{ij} > 0$. The complete set of directed-arc costs is given below in a row-by-row fashion. For completeness, the diagonal elements (self-loops) are set to a dummy value $\langle (100,100,100,100); 1.0 \rangle$ and are never selected because of the constraint $x_{ii} = 0$.

Row 1 (from city 1 – depot):

$\tilde{c}_{12} = \langle (3.5, 5.2, 7.8, 9.1);\ 0.7 \rangle$.

$\tilde{c}_{13} = \langle (2.8, 4.1, 6.3, 8.5);\ 0.6 \rangle$.

$\tilde{c}_{14} = \langle (5.7, 7.3, 9.6, 11.2);\ 0.8 \rangle$.

$\tilde{c}_{15} = \langle (4.2, 6.1, 8.4, 10.7);\ 0.5 \rangle$.

$\tilde{c}_{16} = \langle (7.9, 9.5, 12.3, 14.8);\ 0.9 \rangle$.

$\tilde{c}_{17} = \langle (6.4, 8.2, 10.9, 13.5);\ 0.7 \rangle$.

Row 2 (from city 2):

$\tilde{c}_{21} = \langle (2.7, 4.3, 6.8, 8.9);\ 0.8 \rangle$.

$\tilde{c}_{23} = \langle (5.9, 7.6, 10.1, 12.3);\ 0.6 \rangle$.

$\tilde{c}_{24} = \langle (4.8, 6.7, 9.2, 11.8);\ 0.5 \rangle$.

$\tilde{c}_{25} = \langle (8.6, 10.9, 13.7, 16.2);\ 0.8 \rangle$.

$\tilde{c}_{26} = \langle (7.2, 9.4, 12.1, 14.9);\ 0.7 \rangle$.

$\tilde{c}_{27} = \langle (10.8, 13.2, 16.3, 19.1);\ 0.9 \rangle$.

Row 3 (from city 3):

$\tilde{c}_{31} = \langle (4.1, 5.9, 8.3, 10.6);\ 0.6 \rangle$.

$\tilde{c}_{32} = \langle (6.8, 8.7, 11.4, 13.9);\ 0.7 \rangle$.

$\tilde{c}_{34} = \langle (8.9, 11.2, 14.1, 16.8);\ 0.8 \rangle$.

$\tilde{c}_{35} = \langle (7.5, 9.8, 12.6, 15.3);\ 0.5 \rangle$.

$\tilde{c}_{36} = \langle (11.7, 14.3, 17.5, 20.8);\ 0.9 \rangle$.

$\tilde{c}_{37} = \langle (10.2, 12.9, 16.1, 19.3);\ 0.7 \rangle$.

Row 4 (from city 4):

$\tilde{c}_{41} = \langle (5.6, 7.4, 10.2, 12.7);\ 0.7 \rangle$.

$\tilde{c}_{42} = \langle (8.3, 10.6, 13.7, 16.4);\ 0.8 \rangle$.

$\tilde{c}_{43} = \langle (10.1, 12.8, 16.2, 19.5);\ 0.9 \rangle$.

$\tilde{c}_{45} = \langle(12.9, 15.7, 19.2, 22.8);\ 0.6\rangle$.

$\tilde{c}_{46} = \langle(11.3, 14.1, 17.6, 21.2);\ 0.5\rangle$.

$\tilde{c}_{47} = \langle(15.8, 18.9, 22.7, 26.4);\ 0.8\rangle$.

Row 5 (from city 5):

$\tilde{c}_{51} = \langle(7.2, 9.5, 12.8, 15.6);\ 0.8\rangle$.

$\tilde{c}_{52} = \langle(10.4, 13.2, 16.7, 19.9);\ 0.9\rangle$.

$\tilde{c}_{53} = \langle(12.7, 15.6, 19.3, 22.8);\ 0.5\rangle$.

$\tilde{c}_{54} = \langle(15.3, 18.4, 22.3, 26.1);\ 0.7\rangle$.

$\tilde{c}_{56} = \langle(17.9, 21.2, 25.1, 29.3);\ 0.8\rangle$.

$\tilde{c}_{57} = \langle(16.4, 19.7, 23.6, 27.5);\ 0.6\rangle$.

Row 6 (from city 6):

$\tilde{c}_{61} = \langle\ (9.1, 11.8, 15.3, 18.6);\ 0.9\rangle$.

$\tilde{c}_{62} = \langle(12.6, 15.7, 19.6, 23.2);\ 0.5\rangle$.

$\tilde{c}_{63} = \langle(15.2, 18.6, 22.7, 26.5);\ 0.7\rangle$.

$\tilde{c}_{64} = \langle(18.1, 21.5, 25.8, 29.7);\ 0.8\rangle$.

$\tilde{c}_{65} = \langle(20.7, 24.3, 28.6, 32.9);\ 0.6\rangle$.

$\tilde{c}_{67} = \langle(22.9, 26.7, 30.9, 35.2);\ 0.9\rangle$.

Row 7 (from city 7):

$\tilde{c}_{71} = \langle\ (11.3, 14.2, 18.1, 21.7);\ 0.5\rangle$.

$\tilde{c}_{72} = \langle(14.9, 18.3, 22.8, 26.9); 0.7\rangle$.

$\tilde{c}_{73} = \langle(17.8, 21.4, 25.9, 30.2); 0.8\rangle$.

$\tilde{c}_{74} = \langle(20.9, 24.8, 29.5, 34.1); 0.9\rangle$.

$\tilde{c}_{75} = \langle(23.8, 27.9, 32.7, 37.4); 0.6\rangle$.

$\tilde{c}_{76} = \langle(26.7, 30.9, 35.8, 40.6); 0.5\rangle$.

4.2 | Defuzzified Crisp Cost Matrices

The parametric defuzzification function M_α, defined in *Eq. (15)* of Section 3, was applied to each fuzzy cost $\tilde{c}_{ij}$. The resulting crisp cost matrices are displayed in *Tables 2–7* for α = 0.0, 0.1, 0.2, 0.3, 0.4, 0.5. All off-diagonal entries are rounded to two decimal places. A clear monotonic decrease with increasing α is observed; at α = 0.5, several costs become zero, reflecting the most optimistic attitude (see *Theorem 3*). The diagonal entries, though irrelevant, are left at the dummy value of the crisp equivalent of the self-loop fuzzy cost, which is 200 for α =0.0 and gradually decreases to 100 at α =0.5.

Table 2. Crisp cost matrix for α = 0.0.

Cost	City 1	City 2	City 3	City 4	City 5	City 6	City 7
City 1	200.00	8.96	6.51	13.52	7.35	20.03	13.65
City 2	9.08	200.00	10.77	8.13	19.76	15.26	26.73
City 3	8.67	14.28	200.00	20.40	11.30	28.94	20.47
City 4	12.57	19.60	26.37	200.00	21.18	16.05	33.52
City 5	18.04	27.09	17.60	28.73	200.00	37.40	26.16
City 6	24.66	17.78	29.05	38.04	31.95	200.00	52.06
City 7	16.33	29.02	38.12	49.19	36.54	33.50	200.00

Table 3. Crisp cost matrix for α = 0.1.

Cost	City 1	City 2	City 3	City 4	City 5	City 6	City 7
City 1	180.00	7.70	5.39	11.83	5.86	17.76	11.67
City 2	7.92	180.00	8.95	6.47	17.28	13.05	23.74
City 3	7.20	12.21	180.00	17.83	9.02	25.69	17.53
City 4	10.74	17.13	23.41	180.00	17.62	12.81	29.30
City 5	15.76	24.06	14.06	24.60	180.00	32.69	21.78
City 6	21.89	14.20	24.88	33.26	26.60	180.00	46.26
City 7	13.03	24.84	33.32	43.69	30.43	26.78	180.00

Table 4. Crisp cost matrix for α = 0.2.

Cost	City 1	City 2	City 3	City 4	City 5	City 6	City 7
City 1	160.00	6.43	4.28	10.14	4.39	15.51	9.69
City 2	6.77	160.00	7.15	4.83	14.81	10.86	20.76
City 3	5.75	10.16	160.00	15.27	6.76	22.45	14.59
City 4	8.93	14.67	20.46	160.00	14.07	9.58	25.10
City 5	13.49	21.04	10.52	20.48	160.00	27.98	17.40
City 6	19.13	10.64	20.72	28.49	21.25	160.00	40.46
City 7	9.75	20.68	28.54	38.20	24.32	20.06	160.00

Table 5. Crisp cost matrix for α = 0.3.

Cost	City 1	City 2	City 3	City 4	City 5	City 6	City 7
City 1	140.00	5.15	3.19	8.45	2.92	13.26	7.73
City 2	5.63	140.00	5.35	3.21	12.33	8.67	17.78
City 3	4.30	8.11	140.00	12.71	4.50	19.22	11.66
City 4	7.12	12.21	17.52	140.00	10.53	6.37	20.89
City 5	11.23	18.02	7.00	16.36	140.00	23.29	13.04
City 6	16.38	7.08	16.57	23.73	15.92	140.00	34.66
City 7	6.49	16.52	23.76	32.72	18.23	13.36	140.00

Table 6. Crisp cost matrix for α = 0.4.

Cost	City 1	City 2	City 3	City 4	City 5	City 6	City 7
City 1	120.00	3.87	2.11	6.76	1.45	11.03	5.78
City 2	4.49	120.00	3.56	1.60	9.86	6.49	14.81
City 3	2.86	6.07	120.00	10.16	2.24	16.00	8.73
City 4	5.33	9.76	14.58	120.00	7.01	3.18	16.70
City 5	8.97	15.01	3.50	12.26	120.00	18.61	8.68
City 6	13.63	3.54	12.42	18.97	10.60	120.00	28.87
City 7	3.24	12.38	18.99	27.25	12.14	6.68	120.00

Table 7. Crisp cost matrix for $\alpha = 0.5$.

Cost	City 1	City 2	City 3	City 4	City 5	City 6	City 7
City 1	100.00	2.59	1.05	5.07	0.00	8.80	3.84
City 2	3.36	100.00	1.77	0.00	7.39	4.32	11.84
City 3	1.42	4.04	100.00	7.61	0.00	12.78	5.81
City 4	3.54	7.31	11.65	100.00	3.50	0.00	12.51
City 5	6.72	12.00	0.00	8.16	100.00	13.94	4.33
City 6	10.89	0.00	8.27	14.22	5.30	100.00	23.08
City 7	0.00	8.24	14.23	21.78	6.06	0.00	100.00

As α increases, all off-diagonal costs strictly decrease (or remain zero). The zero entries at $\alpha = 0.5$ correspond to arcs whose fuzzy numbers have $\mu = 0.5$ and for which the crisp value vanishes exactly at the boundary $\alpha = \mu$. In practice, α should be chosen strictly smaller than $\min \mu_{ij}$ to avoid zero costs; here $\alpha = 0.5$ is the theoretical limit and is included only for illustration.

4.3 | Optimal Tours and Sensitivity to α

Each crisp matrix was supplied to the corrected MILP model of Section 2. The solver returned an optimal solution (exit flag = 1) for every α. The solutions were verified to ensure that all non-depot cities are visited exactly once, tour lengths lie within [3], [13], and no subtour exists. The results are summarized in *Table 8*.

Table 8. Optimal tours and total crisp costs for all α values.

α	Total cost	Tour of Salesman 1 (Length)	Tour of Salesman 2 (Length)	Solver Time (s)
0.0	116.6200	1–4–6–2–3–1 (4 cities)	1–5–7–1 (2 cities)	0.8132
0.1	95.6673	1–4–6–2–3–1 (4 cities)	1–5–7–1 (2 cities)	0.7387
0.2	74.7893	1–4–6–2–3–1 (4 cities)	1–5–7–1 (2 cities)	0.9252
0.3	52.7209	1–5–3–1 (2 cities)	1–7–6–2–4–1 (4 cities)	0.7319
0.4	30.7211	1–5–3–1 (2 cities)	1–7–6–2–4–1 (4 cities)	0.9220
0.5	8.8070	1–5–3–1 (2 cities)	1–7–6–2–4–1 (4 cities)	0.7503

The total cost decreases strictly as α rises, empirically confirming *Theorem 2*. The solver times are all under one second, reflecting the small size of the instance and the efficiency of the MILP formulation.

A notable observation is the structural change of the optimal tour set at $\alpha = 0.3$. For conservative attitudes ($\alpha = 0.0, 0.1, 0.2$), the solution consists of one longer tour (1–4–6–2–3–1, 4 cities) and one shorter tour (1–5–7–1, 2 cities). From $\alpha = 0.3$ onward, the pattern reverses: the shorter tour becomes 1–5–3–1 (2 cities), and the longer tour covers the remaining cities (1–7–6–2–4–1, 4 cities). This switch is driven by the differential reduction of arc costs as α increases. For instance, the fuzzy cost $\tilde{c}_{15}$ has height $\mu = 0.5$ and a relatively broad support; its defuzzified value drops from 7.35 at $\alpha = 0.0$ to 2.92 at $\alpha = 0.3$ and eventually to zero at $\alpha = 0.5$. Simultaneously, $\tilde{c}_{53}$ declines to zero as well. Consequently, the cycle 1–5–3–1 becomes increasingly attractive, freeing salesman 2 to serve the remaining four cities.

The auxiliary variables u_i from the MILP solution confirm the strict sequencing. For the solution at $\alpha = 0.0$, the values are:

Tour 1: $u_4 = 1, u_6 = 2, u_2 = 3, u_3 = 4,$

Tour 2: $u_5 = 1, u_7 = 2.$

Both sequences satisfy $u_j = u_i + 1$ for each traversed arc, ensuring that the number of non-depot cities in a tour equals the u-value of its last city. The satisfaction of these sequencing constraints demonstrates that the strict sequencing constraints correctly enforce the minimum tour length k=2 (since $u_3 = 4 \geq 2$ and $u_7 = 2 \geq 2$) and maximum length w=6.

4.4 | Managerial Insights

The parameter α acts as an intuitive, continuous dial that translates the decision-maker's attitude toward risk into concrete routing plans. A small α (e.g., 0.0–0.2) corresponds to a conservative stance: The full spread of the fuzzy numbers is considered, resulting in higher total cost estimates but routes that are likely to remain feasible under worst-case conditions. Larger α values (0.3–0.5) represent an increasingly optimistic outlook, producing lower cost estimates and more aggressive routing choices that exploit the core of the fuzzy numbers.

The abrupt change in the optimal tour configuration at α= 0.3 carries an important practical message: Even a moderate shift in the perceived level of uncertainty can fundamentally alter the optimal operational plan. A logistics manager who slightly relaxes the conservatism of cost estimates may find that a completely different assignment of cities to vehicles becomes preferable. The proposed methodology enables such managers to quantify and visualize the trade-off between cost and robustness, supporting informed decision-making under uncertainty.

Moreover, the analysis highlights the necessity of selecting α strictly below the minimum fuzzy height of all arcs. In the present instance, $\min \mu_{ij} = 0.5$; choosing α = 0.5 yields several zero costs, which may be unrealistic. In practice, one would select α values in the interval $[0, \mu_{\min})$, for example 0.4 or lower, to avoid degenerate costs while still exploring the full spectrum of risk attitudes.

4.5 | Summary of the Numerical Study

The numerical experiments confirm the correctness and practical utility of the proposed two-stage method. The corrected MILP formulation reliably produces feasible, subtour-free solutions that respect all cardinality constraints. The defuzzification function M_α behaves precisely as predicted by the theory: It is continuous, strictly decreasing in α, and produces positive crisp costs for $\alpha < \mu$. The sensitivity analysis reveals that α is not merely a numerical scaling factor but can induce qualitative changes in optimal routing, thereby serving as an effective risk-management lever. The next section concludes the paper and outlines promising directions for future research.

5 | Conclusion

In this paper, a parametric methodology for the single-depot mTSP in a generalized trapezoidal fuzzy environment was presented. The principal achievements are twofold. First, a thorough theoretical investigation of the defuzzification function M_α was carried out. *Theorems 1–4* rigorously establish that the function is continuous, differentiable, strictly decreasing in α, positive for all $\alpha < \mu$, and either convex, linear, or concave depending on the asymmetry of the fuzzy number. These results supply a robust foundation for employing M_α as a reliable ranking function in fuzzy optimization contexts. Second, the function was integrated into a two-stage solution framework. In the first stage, the fuzzy cost matrix is transformed into a crisp matrix for a chosen α; in the second stage, a MILP model with correct subtour elimination and strict sequencing constraints is solved to optimality. The numerical experiments on a 7-city, 2-salesman instance with parameters k=2, w=6, yielded optimal tours for six risk levels $\alpha \in \{0.0, 0.1, \dots, 0.5\}$. The total cost decreased monotonically from 116.62 ($\alpha = 0.0$) to 8.81 ($\alpha = 0.5$), confirming the theoretical monotonicity. Notably, at $\alpha = 0.3$ the optimal tour set changed qualitatively, demonstrating that even moderate shifts in the risk parameter can significantly alter routing decisions. All solutions were obtained in under one second using a standard MILP solver on a desktop computer.

From a managerial perspective, the parameter α offers an intuitive, continuous control for balancing conservatism and optimism. Risk-averse planners may select small α values, resulting in higher cost estimates but more robust routes, while risk-seeking decision-makers can adopt larger α to pursue lower-cost plans.

The ability to generate and compare structurally different optimal tours by simply varying α provides a practical decision-support mechanism for logistics and transportation planning under uncertainty.

Future work can extend the present study in several directions. First, the scalability of the MILP approach may be enhanced by designing specialized exact algorithms or by integrating the defuzzification stage into metaheuristic frameworks (e.g., ACO, GA) for large-scale instances. Second, the use of other types of fuzzy numbers, such as type-2 or spherical fuzzy numbers, could be explored to capture more complex uncertainty structures. Third, additional real-world constraints, such as time windows, heterogeneous fleets, or multiple depots, could be incorporated to broaden the applicability of the method.

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Data Availability

Sufficient data is available upon request.

Conflict of Interest

No potential conflict of interest was reported by the authors.

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